L^p Asymptotic Behavior of Perturbed Viscous Shock Profiles

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Abstract

We investigate the L^p asymptotic behavior $(1 \leq p \leq \infty)$ of a perturbation of a Lax or overcompressive type shock wave solution to a system of conservation law in one dimension. The system of the equations can be strictly parabolic, or have real viscosity matrix (partially parabolic, e.g., compressible Navier–Stokes equations or equations of Magnetohydrodynamics). We use known pointwise Green function bounds for the linearized equation around the shock to show that the perturbation of such a solution can be decomposed into a part corresponding to shift in shock position or shape, a part which is the sum of diffusion waves, i.e., the solutions to a viscous Burger's equation, conserving the initial mass and convecting away from the shock profile in outgoing modes, and another part which is more rapidly decaying in L^p.

1 Introduction

Consider the system of conservation laws with viscosity:

$$u_t + f(u)_x = \nu(B(u)u_x)_x \tag{1.1}$$

with $u \in \mathbb{R}^n$ is the conserved quantity, and ν is a constant measuring transport effects (e.g. viscosity or heat conduction). As we are not considering the vanishing-viscosity limit $\nu \to 0$, we can assume $\nu = 1$. An important

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class of the solutions for (1.1) are the viscous shock wave solutions, i.e., solutions in the form $\bar{u}(x,t) = \bar{u}(x - st)$, where the constant s is the velocity of the shock, and where $\bar{u}$ connects the endstates $u_\pm = \bar{u}(\pm\infty)$. With a simple change of coordinates, we can assume that $s = 0$ (a *stationary* shock solution). $\bar{u}$ is assumed to be an element of a smooth manifold $\{\bar{u}^\delta\}_{\delta\in\mathbb{R}^d}$, which consists of stationary solutions of (1.1) connecting the same endstates u_- and u_+, and $\bar{u} = \bar{u}^0$. Loosely stated, we prove that, with $\tilde{u}$ a solution of (1.1) and a small perturbation of $\bar{u}$, there is a small δ_0 such that

$$\tilde{u}(x,t) - \bar{u}^{\delta_0}(x) = v(x,t) + \varphi(x,t) + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t) \tag{1.2}$$

where

1. $|v(\cdot,t)|_{L^p} \sim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}$. By choosing appropriate δ_0 and the mass carried by φ, we also obtain zero initial mass for v, i.e., $\int_{-\infty}^{+\infty} v(x,0)\,dx = 0$.

2. φ is a summation of convecting diffusion waves, i.e., self similar solutions to the viscous Burger's equation with appropriate coefficients, propagating away from the shock and preserving the initial mass in the outgoing modes, $|\varphi(\cdot,t)|_{L^p} \sim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$, and

3. $|\delta(t)| \sim (1+t)^{-\frac{1}{2}+\epsilon}$ and $\delta(0) = 0$. One can view $\delta(t)$ as indexing the "instantaneous" shock location and shape: employing Taylor's expansion gives us

$$\bar{u}^{\delta_0+\delta(t)} - \bar{u}^{\delta_0} = \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t) + \mathbf{O}(|\delta(t)|^2 e^{-k|x|}) \tag{1.3}$$

   which shows that $\frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$ corresponds to a shift in the shock location and or shape (up to an error of order $|\delta(t)|^2 e^{-k|x|}$, which decays faster than v in any L^p norm).

For the exact definitions and conditions, see the subsequent sections; especially see Theorem 4.14, corollaries 4.15, 4.16 and their counterparts in the real viscosity case, which comprise the main results of this paper.

To prove the above statements, we use (1.2) and initial equations for $\tilde{u}$ and $\bar{u}$ to obtain

$$v_t - Lv = \mathcal{R}(v, v_x, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t), \varphi) \tag{1.4}$$

where L stands for the linearized operator around $\bar{u}$ and $\mathcal{R}$ is a remainder we get applying Taylor's expansion. If $G(x,t;y)$ is the Green function corresponding to $\partial_t - L$, then applying Duhamel's principle yields:

$$
\begin{aligned}
v(x,t) &= \int_{-\infty}^{+\infty} G(x,t;y)v(y,0)dy \\
&\quad + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\mathcal{R}(v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t), \varphi)(y,s)dyds;
\end{aligned}
\tag{1.5}
$$

we try, then, to use a continuous induction to prove the desired rates of decay for v.

The observation that a perturbation of a shock wave solution to (1.1) can be decomposed into a sum of diffusion waves and a more rapidly decaying term is due to T.P. Liu (see **[Liu1, Liu2]**). He proved the result for weak shocks and with the viscosity matrix B being identity matrix. To prove the result, he first constructed an approximate Green function using heavily the weak shock wave assumption and the identity matrix, and then used an elaborate pointwise nonlinear iteration scheme.

We, on the other hand, have already at our disposal the Green function bounds we need. These sharp bounds are the result of a "dynamical system" approach based on Evans function and inverse Laplace transform techniques. This approach began for the strictly parabolic case (**[GZ, ZH]**) and then was extended to many other, more physical, regimes, such as real viscosity case (**[MaZ.1]** – **[MaZ.4]**; see also **[Z1, Z.2]**). In these papers the $L^p, 1 < p \leq \infty$, asymptotic stability and L^1 asymptotic boundedness, of Lax type shock profiles were stated and proved by finding sharp Green function bounds for the linearized equation; some hints have also been given about Green function bounds for the overcompressive case. This approach does not require any assumption of the weakness of the shock profile, and the structural and technical assumptions made about the equation and the wave are rather minimal. However, less information than what Liu's approach yields has been obtained about the behavior of the perturbation.

Using the very same Green distribution bounds, we prove the results which Liu first observed with fewer assumptions: the shock profile can have small or large amplitude, be of Lax or overcompressive type (to our knowledge, this result is the first rather complete result about asymptotic behavior of a perturbed overcompressive shock); the viscosity term $(B(u)u_x)_x$ can be strictly hyperbolic (section 4) or have the block structure of the real viscosity case (section 5). Also no pointwise bounds on initial data are required, only bounds on L^p norm and moments. In return for localization of the initial

data, Liu obtains pointwise bounds for the perturbation. We only assume smallness of the initial perturbation and its moment in some L^P spaces, but then no pointwise information is obtained (we believe, however, that with a similar method and some more work, and with localization of the initial data, we can achieve pointwise bounds similar to those Liu has obtained).

Plan of the paper: In computing the bounds for v using (1.5), we frequently use Young–Hausdorf inequality and L^p norms of the different components of $\mathcal{R}$. However, a term that does not yield the necessary estimates this way occurs and that is when part of the Green function that is like a convecting heat kernel is convoluted against $(\varphi_i^2)_y(y,s)$, with φ_i a diffusion wave convecting at a different speed. Sharp estimation of such terms was first treated by Liu **[Liu1]**. Here, we extract the essential features of his argument, to establish that similar bounds hold whenever the derivative of G along characteristic directions decays more rapidly than G_y: in particular, for the more general Green function terms we consider here. Most of section 2 is to compute pointwise bounds for this part of the calculations. In section 3 we consider the case, already well established (see **[LZe, Kaw1, Kaw2, CL]**), of a strictly parabolic system with a perturbation of a constant state solution. The calculations foreshadow the more difficult case of shock waves. In sections 4 and 5 we consider the perturbation of a shock wave solution in the strictly parabolic case and the real viscosity case, respectively, which are very similar; the main difference is that while in section 4 we use strict parabolicity to establish short time estimates and thereby find good bounds for v_x, in the real viscosity case we have to go through a long list of energy estimates to find the bounds we need for the derivatives. These bounds generalize similar energy estimates obtained in **[MaZ.2, MaZ.4, Z.2, Z.3]**, which in turn generalize the important estimates obtained by Kawashima and others (see **[Kaw]** and references therein) for perturbations of constant states.

2 Some preliminary computations

In this section, we establish some pointwise bounds for the solution $u(x,t)$ of

$$\begin{cases} u_t - u_{xx} = (K(x-t,t)^2)_x & \text{for } t>0 \\ u(x,0) \ = 0 \end{cases} \tag{2.1}$$

Here and elsewhere in this article $K(x,t)$ and $g(x,t)$ both denote the heat kernel: $g(x,t) = K(x,t) = (4\pi t)^{-\frac{1}{2}} e^{\frac{-x^2}{4t}}$

Using Duhamel's principle we obtain from (2.1),

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_{-\infty}^{+\infty} g(x-y, t-s)(K(y-s,s)^2)_y \, dy \, ds \\
&= \int_0^t \int_{-\infty}^{+\infty} g_y(x-y, t-s) K(y-s,s)^2 \, dy \, ds
\end{aligned} \tag{2.2}
$$

The following bounds for u are essential for obtaining L^1 bounds in subsequent sections. They are similar in nature to the bounds given by **[Liu1]**, but the proof given here is different from that of Liu, and is somewhat more general.

Proposition 2.1. *Let $u(x,t)$ be the solution of (2.1) given by (2.2). If $t \geq 1$, then we have*

$$
\begin{aligned}
|u(x,t)| \leq C t^{-\frac{1}{4}} \left(g(x, 4t) + g(x-t, 4t) \right) \\
+ C \chi_{\{\sqrt{t} \leq x \leq t - \sqrt{t}\}} (t^{-1} x^{-\frac{1}{2}} + t^{-\frac{1}{2}} (t-x)^{-1}).
\end{aligned} \tag{2.3}
$$

where χ stands for the indicator function, and C is a constant independent of t and x.
The same result holds if K^2 in (2.1) and (2.2) is replaced with K_x.

Remark 2.2. The result just mentioned is achieved by detecting crucial cancelation in the calculations. In fact, if we replace the integrands in (2.2) by their absolute value, we will obtain the following bounds instead (See **[HZ]**):

$$
|u(x,t)| \leq C \left(g(x, 4t) + g(x-t, 4t) \right) + C \chi_{\{\sqrt{t} \leq x \leq t - \sqrt{t}\}} (x^{-\frac{1}{2}} (t-x)^{-\frac{1}{2}}).
$$

Proof of Proposition 2.1. As $K^2 \sim K_x$, the proof will be stated only for K^2. It would be straightforward to observe that the same argument works for K_x at every step.

We begin the proof by first stating a simple lemma:

Lemma 2.3. *If $0 \leq s \leq \sqrt{t}$, then $e^{\frac{-(x \pm s)^2}{4t}} \leq C e^{\frac{-x^2}{8t}}$ with C independent of t, s and x.*

Proof. The statement of the lemma is equivalent to

$$
\frac{-(x \pm s)^2}{4t} \leq \frac{-x^2}{8t} + D
$$

for some D, which (after some calculation) in its turn is equivalent to $(x \pm 2s)^2 - 2s^2 \geq -8Dt$, which holds for $D > \frac{1}{4}$, since $s^2 < t$. □

In the proof of the proposition, we will, in several places, use the following lemma, which is due to P. Howard [**Ho**].

Lemma 2.4 (P. Howard). *Let $f(\sigma) \geq 0$ be a nonincreasing function on $\mathbb{R}_+$ and $f(0) < \infty$. Assume further that there exist constants $\gamma > 0, \omega > 1$ so that $f(\sigma) \geq \gamma e^{-\frac{a}{2}(1-\frac{1}{\omega})^2\sigma^2}$ on $\mathbb{R}_+$. Then for $a, z > 0$,*

$$\int_0^{+\infty} e^{-a(z-\sigma)^2} f(\sigma)\, d\sigma \leq \frac{C(\omega)}{\sqrt{a}} f\Big(\frac{z}{\omega}\Big).$$

Proof. See page 102 of [**Ho**]. □

Remark 2.5. This result is sharp down to scale $a^{-\frac{1}{2}}$. In practice, this is often augmented with L^∞ bounds obtained by other means. See for example lemma 5 of [**HZ**] or remark 4.17 of this paper.

In what follows we will frequently use the following properties of the heat kernel g, which are easy to prove:

$$\int_{-\infty}^{+\infty} g(x-y,t)g(y,t')dy = g(x,t+t') \tag{2.4}$$

$$|g_x(x,t)| \leq Ct^{-\frac{1}{2}} g(x,2t) \tag{2.5}$$

$$|g_t(x,t)| \leq Ct^{-1} g(x,2t) \tag{2.6}$$

$$|g(x,t)| \leq C\, t^{-\frac{1}{2}}. \tag{2.7}$$

Rewriting (2.2), we have:

$$\begin{aligned} u(x,t) &= \int_0^t \int_{-\infty}^{+\infty} g(x-y,t-s)(g(y-s,s)^2)_y\, dy\, ds \\ &= \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} g(x-y,t-s)(g(y-s,s)^2)_y\, dy\, ds \\ &+ \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x-y,t-s)(g(y-s,s)^2)_y\, dy\, ds \\ &+ \int_{t-\sqrt{t}}^{t} \int_{-\infty}^{+\infty} g(x-y,t-s)(g(y-s,s)^2)_y\, dy\, ds \\ &=: I + II + III. \end{aligned} \tag{2.8}$$

(I) and (III) are easy to estimate:

$$|I| = \left| \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} g(x-y, t-s)(g(y-s,s)^2)_y \, dy \, ds \right| \tag{2.9}$$

$$= \left| \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} g_y(x-y, t-s) g(y-s,s)^2 \, dy \, ds \right| . \tag{2.10}$$

By (2.5) and (2.7), the above is,

$$\leq C \int_0^{\sqrt{t}} \int_{-\infty}^{+\infty} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} g(x-y, 2(t-s))\, g(y-s, 2s) \, dy \, ds$$

which is, by (2.4),

$$\leq C \int_0^{\sqrt{t}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} g(x-s, 2t) \, ds.$$

Now, using lemma 2.3, the above is

$$\begin{aligned} &\leq \quad C\, g(x, 4t) \int_0^{\sqrt{t}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \, ds \\ &\leq \quad C t^{-\frac{1}{2}} g(x, 4t) \int_0^{\sqrt{t}} s^{-\frac{1}{2}} \, ds \\ &\leq \quad C t^{-\frac{1}{4}} g(x, 4t). \end{aligned}$$

Part (III) in (2.8) can be handled similarly.

The more difficult part is part (II) of (2.8):

$$II = \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x-y, t-s)(g(y-s,s)^2)_y \, dy \, ds. \tag{2.11}$$

In order to estimate (II), let us write $g = g(x, \tau)$. We have then,

$$\begin{aligned} g(x-y,t-s)(g(y-s,s)^2)_y \; &= (g(x-y,t-s)g(y-s,s)^2)_s \\ &- \; g_\tau(x-y,t-s)g(y-s,s)^2 \\ &+ \; g(x-y,t-s)(g^2)_\tau(y-s,s). \end{aligned} \tag{2.12}$$

We will do the estimates piece by piece.

The first part of (2.12) can be estimated as follows.

$$\int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} (g(x-y,t-s)g(y-s,s)^2)_s \, dy \, ds \tag{2.13}$$

$$= \int_{-\infty}^{+\infty} g(x-y,\sqrt{t})\, g(y-t+\sqrt{t}, t-\sqrt{t})^2 \, dy \tag{2.14}$$

$$- \int_{-\infty}^{+\infty} g(x-y,t-\sqrt{t})\, g(y-\sqrt{t},\sqrt{t})^2 \, dy. \tag{2.15}$$

Using (2.4) and (2.7), we will have:

$$\int_{-\infty}^{+\infty} g(x-y,\sqrt{t})\, g(y-t+\sqrt{t}, t-\sqrt{t})^2 \, dy \le Ct^{-\frac{1}{2}} g(x-t+\sqrt{t}, t),$$

and

$$\int_{-\infty}^{+\infty} g(x-y,t-\sqrt{t})\, g(y-\sqrt{t},\sqrt{t})^2 \, dy \le Ct^{-\frac{1}{4}} g(x-\sqrt{t}, t),$$

but by lemma 2.3

$$g(x-\sqrt{t},t) \le g(x,2t)$$
$$g(x-t+\sqrt{t},t) \le g(x-t,2t).$$

These terms fit in the right hand side of (2.3).

For the other parts in (2.12), we use (2.6) to obtain:

$$\begin{aligned}
&\left| \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g_\tau(x-y,t-s)g(y-s,s)^2 \, dy \, ds \right| \\
&\le \int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\
&= \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\
&+ \int_{\frac{t}{2}}^{t-\sqrt{t}} s^{-\frac{1}{2}}(t-s)^{-1} g(x-s,2t) ds \\
&=: \mathcal{A} + \mathcal{B}.
\end{aligned} \tag{2.16}$$

If $x \le \sqrt{t}$, then

$$\begin{aligned}
\mathcal{A} &\le C\, t^{-1} g(x-\sqrt{t},2t) \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}} \, ds \\
&\le C\, t^{-\frac{1}{2}}\, g(x-\sqrt{t},2t) \\
&\le Ct^{-\frac{1}{2}} g(x,4t).
\end{aligned} \tag{2.17}$$

Similarly when $x \geq t - \sqrt{t}$, we obtain $\mathcal{A} \leq C t^{-\frac{1}{2}} g(x-t, 4t)$.

Now for $\sqrt{t} \leq x \leq \frac{t}{2}$ we have:

$$\mathcal{A} = \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}} (t-s)^{-1} g(x-s, 2t) ds \tag{2.18}$$

$$\leq t^{-1} \int_{\sqrt{t}}^{\frac{t}{2}} s^{-\frac{1}{2}} g(x-s, 2t) ds \tag{2.19}$$

$$= C\, t^{-1} \int_{\sqrt{t}}^{\frac{t}{2}} \left(\frac{s}{t}\right)^{-\frac{1}{2}} e^{-\frac{t}{8}(\frac{x}{t} - \frac{s}{t})^2} \frac{ds}{t} \tag{2.20}$$

$$= C\, t^{-1} \int_{\frac{1}{\sqrt{t}}}^{\frac{1}{2}} \sigma^{-\frac{1}{2}} e^{-\frac{t}{8}(\frac{x}{t} - \sigma)^2}\, d\sigma \tag{2.21}$$

Now we use Howard's lemma, lemma 2.4, to find that (2.21) is indeed,

$$\leq C\, t^{-\frac{3}{2}} (\frac{x}{t})^{-\frac{1}{2}} \tag{2.22}$$

$$= C\, t^{-1} x^{-\frac{1}{2}}. \tag{2.23}$$

If, on the other hand, $\frac{t}{2} \leq x \leq t - \sqrt{t}$, then clearly $\mathcal{A}$ is majorized by the value already computed for $x = \frac{t}{2}$, of

$$\mathcal{A} \leq C t^{-\frac{3}{2}} \leq C\, t^{-1} x^{-\frac{1}{2}}, \tag{2.24}$$

also acceptable.

Part $\mathcal{B}$ in (2.16) can be estimated similarly.

Now remains the last part of (2.12), i.e.,

$$\left| \int_{\sqrt{t}}^{t-\sqrt{t}} \int_{-\infty}^{+\infty} g(x-y, t-s) (g^2)_\tau (y-s, s)\, dy\, ds \right|,$$

which can easily be shown to be

$$\leq \int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{3}{2}} g(x-s, 2t)\, ds. \tag{2.25}$$

If $x \leq \sqrt{t}$, then (2.25) is

$$\leq C\, g(x - \sqrt{t}, 2t) \int_{\sqrt{t}}^{t-\sqrt{t}} s^{-\frac{3}{2}}\, ds$$

$$\leq C\, t^{-\frac{1}{4}}\, g(x-\sqrt{t}, 2t)$$
$$\leq C\, t^{-\frac{1}{4}}\, g(x, 4t).$$

For $x \geq t-\sqrt{t}$ we use a similar method.

For $\sqrt{t} \leq x \leq t-\sqrt{t}$, we use a similar method to what we used for the previous case, and again invoke the result in lemma 2.4 to conclude that (2.25) $\leq x^{-\frac{3}{2}}$
This completes our proof. □

As an straightforward consequence to the above proposition we have:

Corollary 2.6. *For $u(x,t)$ the solution of (2.1) given by (2.2), and for $t \geq 1$, we have:*

$$|u(\cdot,t)|_{L^p} \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}, \tag{2.26}$$

where $|\cdot|_{L^p}$ stands for the norm in $L^p(\mathbb{R})$, $1 \leq p \leq +\infty$, and C is a constant. The same result holds if K^2 in (2.1) and (2.2) is replaced with K_x.

Bounds obtained in corollary (2.6) will be used when working to prove L^p bounds in the system case. In that setting, however, we will usually have a convecting diffusion wave instead of $K(x,t)$ in the source, and part of a Green function, convecting at a different speed, in place of $g(x,t)$ in (2.2). The following result deals with those cases.

Corollary 2.7. *In (2.2), if we replace $K(y-at,t)$ with $\phi(y,t)$, and $g(x-y,t)$ with $G(x,t;y)$ in (2.2), we will have similar bounds for $u(x,t)$ obtained in proposition (2.1) and corollary (2.6), provided ϕ and G satisfy the following bounds:*

$$|G(x,t;y)| \leq Cg(x-y-at, \beta t), \tag{2.27}$$

$$|G_y(x,t;y)| \leq Ct^{-\frac{1}{2}}g(x-y-at, 2\beta t), \tag{2.28}$$

$$|G_t(x,t;y)| \leq Ct^{-1}g(x-y-at, 2\beta t), \tag{2.29}$$

$$|\phi(x,t)| \leq Cg(x-bt, \beta t), \tag{2.30}$$

$$|\phi_y(x,t)| \leq Ct^{-\frac{1}{2}}g(x-bt, 2\beta t), \tag{2.31}$$

$$|\phi_t(x,t;y)| \leq Ct^{-1}g(x-bt, 2\beta t), \tag{2.32}$$

for some $a \neq b$, and some constants $C, \beta > 0$.

Proof. A review of the proofs just carried out will show that the above bounds are the only ones used in the proof. Hence everything works in the same way as before. $\square$

Example 2.8.

$$\phi(x,t) = \frac{(e^{m/2\sqrt{\beta}} - 1)t^{-\frac{1}{2}} e^{\frac{-x^2}{4\beta t}}}{\frac{2\sqrt{\pi}}{\sqrt{\beta}} + (e^{m/2\sqrt{\beta}} - 1)\int_{\frac{x}{\sqrt{4\beta t}}}^{+\infty} e^{-\xi^2} d\xi}$$

solves

$$\begin{cases} \phi_t - \beta\phi_{xx} = -(\phi^2)_x & \textit{for} \quad t > 0 \\ \phi(x,0) = m\delta_0 & \qquad\quad t = 0, \end{cases} \tag{2.33}$$

where δ_0 stands for the Dirac distribution, and $m = \int_{-\infty}^{+\infty} \phi(x,t)\, dx$.
The function ϕ satisfies the inequalities in (2.7) (see **[Liu2]**). Also one can easily see that if one puts ϕ_x in place of ϕ^2 in above argument, then again one will obtain similar results, as $\phi_x \sim \phi^2$. This function ϕ will be the prototype of the *diffusion waves*, which we are going to define and use in the next sections.

Now that we are in the mood of working with the heat kernels, let us state some lemmas that we will need when dealing with systems. The proofs are easy and left to the reader.

Lemma 2.9. *If $a_1 \neq a_2$ and $\beta_1, \beta_2 > 0$, then $|K(x - a_1 t, \beta_1 t)K(x - a_2 t, \beta_2 t)|_{L^p} \leq Ce^{-\eta t}$, for some $\eta > 0$.*
The same result holds if one replaces K with ϕ from example (2.8).

And the following lemmas, which will be needed for shock wave cases:

Lemma 2.10. *Assume $a > 0$ (respectively, $a < 0$), $h(x)$ is a bounded function and $h(x) = \mathbf{O}(e^{-|x|})$ as $x \to +\infty$ (respectively, as $x \to -\infty$). Then $|h(x)K(x - at, t)|_{L^p} = \mathbf{O}(e^{-\eta t})$ for some $\eta > 0$.*

Lemma 2.11. *Assume a, b are both of the same sign, then*

$$\int_0^t \int_{-\infty}^{+\infty} g(y + a(t-s), t-s)g(y - bs, s)^2 dy\, ds = \mathbf{O}(e^{-\eta t})$$

for some $\eta > 0$; If a and b are of the different sign then,

$$\int_0^t \int_{-\infty}^{+\infty} g(y + a(t-s), t-s)g(y - bs, s)^2 dy\, ds = \mathbf{O}(t^{-\frac{1}{2}}).$$

Lemma 2.12. *For $t > 0$ and for $a \neq 0$,*

$$\int_{-\infty}^{+\infty} g(y+at,t)e^{(-|y|)}dy = \mathbf{O}(t^{-\frac{1}{2}}e^{-\eta t}).$$

3 System of conservation laws with constant background solution

Now consider the system of conservation laws

$$\tilde{u}_t + f(\tilde{u})_x = (B(\tilde{u})\tilde{u}_x)_x \tag{3.1}$$

with the solution $\tilde{u} = \tilde{u}(x,t) \in \mathbb{R}^n$, a perturbation of the constant background solution $\bar{u} \equiv \bar{u}_0$.

Let $u = \tilde{u} - \bar{u}$, and use Taylor's expansion to obtain

$$u_t + Au_x - Bu_{xx} = -(\Gamma(u,u))_x + \mathbf{O}(|u|^3)_x + \mathbf{O}(|u||u_x|)_x \tag{3.2}$$

where $A = df(\bar{u})$, $B = B(\bar{u})$ and $\Gamma = \frac{1}{2}d^2f(\bar{u})$.

Some basic assumptions have to be made: we assume $f, B \in C^3$, $df(\bar{u})$ is strictly hyperbolic, $Re\,\sigma(B) > 0$ and finally stability criterion of Majda and Pego [**Kaw, MP**]: $Re\,\sigma(-ikA - k^2B) < -\theta k^2$ for all real k and some $\theta > 0$.

Let $a_1, \cdots, a_n$ be the eigenvalues of A (necessarily disjoint by the strict hyperbolicity of A), with corresponding right eigenvectors $r_1, \cdots, r_n$, and left eigenvectors $l_1, \cdots, l_n$, normalized so that $l_i \cdot r_j = \delta_{ij}$. Define Γ^i_{jk} and b^i_j to be the constant coefficients satisfying

$$\Gamma(r_j, r_k) = \sum_{i=1}^{n} \Gamma^i_{jk} r_i, \quad Br_j = \sum_{i=1}^{n} b^i_j r_i \tag{3.3}$$

and set $\beta_i = b^i_i$ and $\gamma_i = \Gamma^i_{ii}$ (notice that it follows from our assumptions about A and B that $\beta_i > 0$). Define *diffusion wave* in r_i direction: $\varphi^i(x,t)$ to be the solution of

$$\begin{cases} \varphi^i_t + a_i\varphi^i_x - \beta_i\varphi^i_{xx} = -\gamma_i(\varphi^{i2})_x & \text{for} \quad t > -1 \\ \varphi^i(x,-1) = m_i\delta_0 & \qquad\quad t = -1 \end{cases} \tag{3.4}$$

where δ_0 is the Dirac distribution, and m_i is the amount of mass $\int_{-\infty}^{+\infty} u(x,0)\,dx$ in the direction r_i. See Example (2.8). Assuming $m \leq E_0$ we will have:

$$\left|\frac{\partial^n}{\partial x^n}\varphi^i(\cdot,t)\right|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{n}{2}}, \tag{3.5}$$

i.e., φ^i has L^p bounds like a heat kernel. It is not difficult to observe that φ acts like a convecting heat kernel; especially for our interest is the fact that it satisfies the bounds (2.30)-(2.32). Finally set

$$\varphi = \sum_{i=1}^{n} \varphi^i r_i.$$

Let $v := u - \varphi$, hence

$$\tilde{u} = \bar{u} + \varphi + v. \tag{3.6}$$

Notice that

$$\int_{-\infty}^{+\infty} v_0 dx = 0. \tag{3.7}$$

Set $V_0(x) := \int_{-\infty}^{x} v_0 dx$.

Lemma 3.1. $|V_0|_{L^1} = \int_{-\infty}^{+\infty} |V_0| dx \leq |xv(x,0)|_{L^1}$, *assuming that the latter quantity is bounded.*

Proof. We can assume, without loss of generality, that v_0 is scalar, i.e., $v_0(x) \in \mathbb{R}$. If $x_0 < 0$, then $|V_0(x_0)| = |\int_{-\infty}^{x_0} v_0(x)dx| \leq \frac{1}{x_0}\int_{-\infty}^{x_0} |x||v_0(x)|dx$, hence $|x_0 V_0(x)| \leq \int_{-\infty}^{x_0} |x||v_0(x)|dx$, which approaches 0, as $x_0 \to -\infty$. Likewise for $x_0 > 0$, we have $V_0(x_0) = \int_{-\infty}^{x_0} v_0(x)dx = -\int_{x_0}^{-\infty} v_0(x)dx$ (because of (3.7)), and a similar argument shows that $|x_0 V_0(x)|$ approaches 0 as $x_0 \to +\infty$. Now assume that (α_1, α_2) is an interval on which V_0 does not change sign (suppose, without loss of generality, it is positive on this interval), and $V_0(\alpha_1) = 0$ and $V_0(\alpha_2) = 0$. Then $\alpha_1 V_0(\alpha_1) = 0$ and $\alpha_2 V_0(\alpha_2) = 0$ (in the case α_1 or α_2 is $\pm\infty$, the aforementioned argument would work). In this interval, then, we will have

$$\int_{\alpha_1}^{\alpha_2} |V_0(x)dx| = \int_{\alpha_1}^{\alpha_2} V_0(x)dx$$

which,by integration by parts, is equal to

$$\alpha_2 V_0(\alpha_2) - \alpha_1 V_0(\alpha_1) - \int_{\alpha_1}^{\alpha_2} xv_0(x)dx \leq \int_{\alpha_1}^{\alpha_2} |xv_0(x)|dx.$$

Now take summation over all the intervals in the aforesaid form. □

Substituting u with $v+\varphi$ in (3.2), we get:

$$\begin{aligned}
& v_t + Av_x - Bv_{xx} \\
& = -(\varphi_t + A\varphi_x - B\varphi_{xx} + \Gamma(\varphi,\varphi)_x) \\
& + \mathbf{O}(|v|^2 + |\varphi||v| + |(\varphi+v)(\varphi+v)_x| + |\varphi+v|^3)_x \\
& =: \Psi(x,t) + \mathcal{F}(v,\varphi)_x.
\end{aligned} \tag{3.8}$$

Using (3.4) and (3.3), we get

$$\begin{aligned}
\Psi(x,t) &= -\left(\varphi_t + A\varphi_x - B\varphi_{xx} + \Gamma(\varphi,\varphi)_x\right) \\
&= -\sum_{i=1}^{n}\sum_{j\neq k}\Gamma^i_{jk}(\varphi^j\varphi^k)_x r_i - \sum_{i\neq j}\Gamma^i_{jj}(\varphi^j)^2_x r_i + \sum_{i\neq j} b^i_j \varphi^j_{xx} r_i.
\end{aligned} \tag{3.9}$$

(The whole point is that, this way, we get rid of the terms $(\varphi^i)^2_x r_i$ and $\varphi^i_{xx} r_i$.)

The Green function for the linear part of (3.8), i.e., for $v_t + Av_x - Bv_{xx}$, is

$$G(x,t;y) = \sum_{i=1}^{n}(4\pi t)^{-\frac{1}{2}} e^{\frac{(x-y-a_i t)^2}{4\beta_i t}} r_i l_i^t + R(x,t;y),$$

with the remainder R,

$$R(x,t;y) = \mathbf{O}((1+t)^{-1}\sum_{i=1}^{n} e^{\frac{(x-y-a_i t)^2}{Mt}})$$

(for proof see [**LZe**]).

Using Duhamel's principle,

$$\begin{aligned}
v(x,t) = & \int_{-\infty}^{+\infty} G(x,t;y)v_0(y)dy \\
& + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)(\mathcal{F}(v,\varphi)_y(y,s) + \Psi(y,s))dy\, ds \\
& = \int_{-\infty}^{+\infty} G_y(x,t;y)V_0(y)dy \\
& + \int_0^t \int_{-\infty}^{+\infty} G_y(x,t-s;y)\mathcal{F}(v,\varphi)(y,s)dy\, ds \\
& + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\Psi(y,s)ds.
\end{aligned} \tag{3.10}$$

Theorem 3.2. *Assume the above setting, $u = v + \varphi$ and assume that $|u_0|_{L^1}$, $|u_0|_{L^\infty}$, $|xu_0(x)|_{L^1} \le E_0$, for sufficiently small E_0 (these inequalities translate into similar ones for v_0 and φ_0). Then,*

$$|v(\cdot,t)|_p \le C\,E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}} \tag{3.11}$$

for some constant C.

Proof. Let

$$\zeta(t) := \sup_{0\le s\le t,\,1\le p\le\infty} |v(\cdot,s)|_{L^p}(1+s)^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{4}}, \tag{3.12}$$

i.e., $|v(\cdot,s)|_{L^p} \le (1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}\zeta(t)$, and in particular $|v(\cdot,s)|_{L^\infty} \le (1+s)^{-\frac{3}{4}}\zeta(t)$. The goal is to show:

$$\zeta(t) \le C(E_0 + \zeta(t)^2).$$

But then, if E_0 is sufficiently small, this implies that $\zeta(t) \le 2CE_0$, and that is what we are looking for.

Obviously $|G_y|_{L^p} \le t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$. Also $|\varphi(\cdot,t)|_{L^p} \le E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}$ and $|\varphi_x(\cdot,t)|_{L^p} \le E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$. We need to find some bounds for v_x, and that is the subject of the following lemma:

Lemma 3.3. *Given above setting, we will have:*

$$|v_x(\cdot,t)|_{L^p} \le \begin{cases} C(|v(\cdot,t-1)|_{L^p} + |\mathcal{M}(\varphi(\cdot,t)|_{L^p}), & \text{for } t \ge 1 \\ C(t^{-\frac{1}{2}}|v_0|_{L^p}| + |\mathcal{M}(\varphi(\cdot,t)|_{L^p}), & \text{for } t \le 1. \end{cases} \tag{3.13}$$

where $\mathcal{M}(\varphi) := -\varphi_t + (B(\bar u+\varphi)\varphi_x)_x - (f(\bar u+\varphi))_x$ and consequently

$$|v_x(\cdot,t)|_{L^p} \le \begin{cases} C(E_0+\zeta(t))t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}} & \text{for } t \ge 1 \\ C(E_0+\zeta(t))t^{-\frac{1}{2}}, & \text{for } t \le 1. \end{cases} \tag{3.14}$$

Proof.

$$\tilde u_t + f(\tilde u)_x = (B(\tilde u)\tilde u_x)_x$$

implies

$$v_t + \varphi_t + f(\bar u+\varphi+v)_x = (B(\bar u+\varphi+v)(\bar u_x+\varphi_x+v_x))_x.$$

Therefore,

$$v_t + (f(\bar u+\varphi+v) - f(\bar u+\varphi))_x - ((B(\bar u+\varphi+v) - B(\bar u+\varphi))(\bar u_x+\varphi_x))_x$$

$$
\begin{aligned}
&-(B(\bar u+\varphi+v)v_x)_x\\
&=-\varphi_t-f(\bar u+\varphi)_x+(B(\bar u+\varphi)\varphi_x)_x\\
&=:\mathcal{M}(\varphi).
\end{aligned}
$$

From here we can use short time estimates described in [**ZH**], section 11, to achieve inequality (3.13) (see the argument in lemma 4.13). (3.14) follows immediately, if we notice that, by definition of φ, we have $\mathcal{M}(\varphi) = \mathbf{O}(|\varphi||\varphi_x|)$. $\square$

Returning to the proof of the theorem, whenever $0 \le s \le t$, then,

$$
|\mathcal{F}(v,\varphi)(\cdot,s)|_{L^p} \le C(E_0+\zeta^2(t))s^{-\frac12(1-\frac1p)-\frac34}, \tag{3.15}
$$

(we used $E_0 \le 1$ and so $E_0^2, E_0^3 \le E_0$ and $E_0\zeta(t) \le \frac12 E_0^2 + \frac12\zeta^2(t)$).

When $t \le 1$, then

$$
\begin{aligned}
|v(\cdot,t)|_{L^p} &\le C|v_0|_{L^p}\\
&\quad+\int_0^t |G_y|_{L^1}|\mathcal{F}(v,\varphi)|_{L^p}(s)ds\\
&\quad+\int_0^t (|G|_{L^1}||\psi(y,s)|_{L^p}(s)ds\\
&\le CE_0+(E_0+\zeta(t)^2)\int_0^t (t-s)^{-\frac12}s^{-\frac12}ds + CE_0\int_0^t (1+s)^{\frac12}\\
&\le C(E_0+\zeta(t)^2) \le C(E_0+\zeta(t)^2)(1+t)^{-\frac12(1-\frac1p)-\frac14}.
\end{aligned} \tag{3.16}
$$

For $t \ge 1$ we use again Haussdorf-Young inequality to obtain:

$$
\begin{aligned}
&\left|\int_{-\infty}^{+\infty} G_y(x,t;y)V_0(y)dy\right|_{L^P}\\
&\le |V_0|_{L^1}|G_y|_{L^p} \le CE_0 t^{-\frac12(1-1/p)-\frac12},
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
&|\int_0^t \int_{-\infty}^{+\infty} G_y(x,t-s;y)\mathcal{F}(v,\varphi)(y,s)dy\,ds|_{L^p} \\
&\quad \le \int_0^{t/2} |G_y|_{L^p} |\mathcal{F}(v,\varphi)(\cdot,s)|_{L^1}\,ds \\
&\qquad + \int_{t/2}^t |G_y|_{L^1} |\mathcal{F}(v,\varphi)(\cdot,s)|_{L^p}\,ds \\
&\quad \le C(E_0+\zeta(t)^2)(\int_0^{t/2} (t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}} s^{-\frac{3}{4}}ds \\
&\qquad + \int_{t/2}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(1-1/p)-\frac{3}{4}}ds) \\
&\quad \le C(\zeta_0+\zeta(t)^2)t^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}. \\
&\quad \le 2C(\zeta_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}.
\end{aligned}
\tag{3.18}
$$

It remains only to deal with the term Ψ in (3.8), which, by (3.9), includes the terms in the form $\Gamma^i_{jk}(\varphi^j\varphi^k)_x r_i$ for $j \neq k$, and $\Gamma^i_{jj}(\varphi^j)^2_x r_i$ and $b^i_j \varphi^j_{xx} r_i$ for $i \neq j$.

Lemma 2.9 takes care of the terms in the form $(\varphi^j\varphi^k)$ for $j \neq k$, as strict hyperbolicity of A implies $a_j \neq a_k$., hence giving us

$$
|(\varphi^j\varphi^k)|_{L_p} \le CE_0^2 e^{-\eta s} \le CE_0(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{4}}. \tag{3.19}
$$

This then will be treated similar to the way (3.15) is treated in (3.18).

For other terms, we need to estimate

$$
\left| \int_0^t \int_{-\infty}^{+\infty} G(x,t;y)(\varphi^j)^2_x r_i \right|_{L^p}
$$

for $i \neq j$. As the remainder R in G is small enough, the only part of concern would be:

$$
\left| \int_0^t \int_{-\infty}^{+\infty} (4\pi t)^{-\frac{1}{2}} e^{\frac{(x-y-a_i t)^2}{4\beta_i t}} (\varphi^j)^2_x \right|_{L^p}
$$

for $i \neq j$. Here we have a heat kernel convecting at the speed a_i convoluted against a diffusion wave φ^j which is similar to ϕ in Example (2.8), but convecting at the speed a_j. Corollary (2.7), then, implies that the above term would be less than $CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}$.
The term $b^i_j\varphi^j_{xx}r_i$ can be treated similarly. This finishes the proof. $\square$

Remark 3.4. In the scalar case, we get rid of the terms in the form of φ^2 all together, since $i = j = 1$ is the only possibility. Hence it can readily be seen that, in the scalar case, we would obtain, using this argument, the decay rates: $|v(\cdot,t)|_{L^p} \sim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}+\epsilon}$, for ϵ arbitrarily small **[Liu1]**.

4 Strictly parabolic cases with a viscous shock solution

We now focus on a shock wave solution of the system of viscous conservation laws

$$\tilde{u}_t + f(\tilde{u})_x = (B(\tilde{u})\tilde{u}_x)_x, \tag{4.1}$$

where $f \in \mathbb{R}^n$ and $B(\tilde{u}) \in \mathbb{R}^{n\times n}$, and $\tilde{u} \in \mathbb{R}^n$ is a perturbation of (without loss of generality) a stationary viscous shock solution

$$\bar{u} = \bar{u}(x), \lim_{x\to\pm\infty} \bar{u}(x) =: u_\pm, \tag{4.2}$$

i.e., $\bar{u}$ solves

$$\bar{u}' = B(\bar{u})^{-1}(f(\bar{u}) - f(u_-)). \tag{4.3}$$

Following **[ZH, Z1]**, we make assumptions $(\mathcal{H})$ below.

Assumptions $(\mathcal{H})$:

$(\mathcal{H}0)$ $f, B \in C^3$.

$(\mathcal{H}1)$ $Re\, \sigma(B) > 0$.

$(\mathcal{H}2)$ $\sigma(df(u_\pm))$ real, distinct, and nonzero.

$(\mathcal{H}3)$ $Re\, \sigma(-ikdf(u_\pm) - k^2B(u_\pm)) < -\theta k^2$ for all real k, some $\theta > 0$.

$(\mathcal{H}4)$ All set of the stationary solutions near $\bar{u}$ of (4.1)-(4.2), connecting the same values $u_\pm$ forms a smooth manifold $\{\bar{u}^\delta\}, \delta \in \mathbb{R}^\ell, \bar{u}^0 = \bar{u}$. Moreover the stable manifold of u_- and the unstable manifold of u_+ (with respect to (4.3)) are transverse.

Condition $(\mathcal{H}3)$ is the *stable viscosity matrix* criterion of Majda and Pego, corresponding to linearized stability of the constant solutions $u \equiv u_\pm$ **[MP, Kaw]** (clearly necessary for stability of $\bar{u}(\cdot)$ of the type we seek, see further discussion (**[ZH]**, pp. 746, 767, and 774–775). Note that condition $(\mathcal{H}4)$ is the condition $H4$ of **[ZH]**, plus the assertion that the shock is of "standard" or "pure" type (see **[ZH]**, section 10). This implies that we

have $n+\ell$ incoming characteristics, entering the shock, hence $n-\ell$ outgoing modes, i.e., eigenvalues are in the form:

$$a_1^- < \cdots < a_{p-1}^- < 0 < a_p^- < \cdots < a_n^-, \tag{4.4}$$

and

$$a_1^+ < \cdots < a_{p+\ell-1}^+ < 0 < a_{p+\ell}^+ < \cdots < a_n^+, \tag{4.5}$$

where $a_i^\pm$ denote the (ordered) eigenvalues of $df(u_\pm)$. If $\ell = 1$, we have a Lax type shock wave, in which case there are $n-1$ outgoing modes (corresponding to $a_i^\pm \gtrless 0$) and $n+1$ incoming modes (corresponding to $a_i^\pm \lessgtr 0$). If $\ell > 1$, we have an overcompressive shock, with $n-\ell$ outgoing modes and $n+\ell$ incoming modes. For further discussion see **[ZH]**.

The following Lemma proved in **[MP]** asserts that $u_\pm$ are hyperbolic also in the ODE sense (for an alternative proof, see Remark 2.3 in section 2). This implies exponential approach of $\bar{u}^\delta$ to its asymptotic states at $x = \pm\infty$, a fact that will be crucial in our subsequent analysis. See **[MP]** and also **[ZH]** for proofs.

Lemma 4.1. *Given $(\mathcal{H}0) - (\mathcal{H}3)$, the stable/unstable manifolds of $df(u_\pm)$ and $B(u_\pm)^{-1}df(u_\pm)$ have equal dimensions. In particular, $B(u_\pm)^{-1}df(u_\pm)$ has no center manifold.*

Corollary 4.2. *Given $(\mathcal{H}0) - (\mathcal{H}4)$, solutions $\bar{u}^\delta$ of (4.3) are in C^4, satisfying*

$$D_x^j D_\delta^i (\bar{u}^\delta(x) - u_\pm) = \mathbf{O}(e^{-\alpha|x|}), \quad \alpha > 0,\ 0 \le j \le 4,\ i = 0, 1,$$

as $x \to \pm\infty$

Linearizing about $\bar{u}(\cdot)$ gives:

$$v_t = Lv := -(Av)_x + (Bv_x)_x, \tag{4.6}$$

with

$$B(x) := B(\bar{u}(x)), \quad A(x)v := df(\bar{u}(x))v - dB(\bar{u}(x))v\bar{u}_x. \tag{4.7}$$

Denoting $A^\pm := A(\pm\infty)$, $B^\pm := B(\pm\infty)$, and considering corollary 4.2, it follows that

$$|A(x) - A^-| = \mathbf{O}(e^{-\eta|x|}), \quad |B(x) - B^-| = \mathbf{O}(e^{-\eta|x|}) \tag{4.8}$$

as $x \to -\infty$, for some positive η. Similarly for A^+ and B^+, as $x \to +\infty$. Also $|A(x) - A^{\pm}|$ and $|B(x) - B^{\pm}|$ are bounded for all x.

Define the *(scalar) characteristic speeds* $a_1^{\pm} < \cdots < a_n^{\pm}$ (as above) to be the eigenvalues of $A^{\pm}$, and the left and right *(scalar) characteristic modes* $l_j^{\pm}$, $r_j^{\pm}$ to be corresponding left and right eigenvectors, respectively (i.e., $A^{\pm} r_j^{\pm} = a_j^{\pm} r_j^{\pm}$, etc.), normalized so that $l_j^+ \cdot r_k^+ = \delta_k^j$ and $l_j^- \cdot r_k^- = \delta_k^j$. Following Kawashima [**Kaw**], define associated *effective scalar diffusion rates* $\beta_j^{\pm} : j = 1, \cdots, n$ by relation

$$\begin{pmatrix} \beta_1^{\pm} & & 0 \\ & \vdots & \\ 0 & & \beta_n^{\pm} \end{pmatrix} = \text{diag } L^{\pm} B^{\pm} R^{\pm}, \tag{4.9}$$

where $L^{\pm} := (l_1^{\pm}, \ldots, l_n^{\pm})^t$, $R^{\pm} := (r_1^{\pm}, \ldots, r_n^{\pm})$ diagonalize $A^{\pm}$.

Let

$$G(x, t; y) := e^{Lt} \delta_y(x) \tag{4.10}$$

be the Green's function associated with $(\partial_t - L)$. Then, the relevant linearized theory can be summarized in the following two propositions, proved in [**ZH**].

Proposition 4.3. *Given ($\mathcal{H}$), necessary conditions for L^p-linearized orbital stability, $p > 0$, of $\bar{u}(\cdot)$ with respect to perturbations $v_0 \in C_0^{\infty}$ are:*
Assumptions ($\mathcal{D}$):

($\mathcal{D}1$) *L has no (L^2, without loss of generality) eigenvalues in $\{Re\lambda \geq 0\} \setminus \{0\}$.*

($\mathcal{D}2$) *$\{r^{\pm}; a^{\pm} \gtrless 0\} \cup \{\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^{\delta}}{\partial \delta_i} dx; i = 1, \cdots, \ell\}$ is a basis for $\mathbb{R}^n$, with $\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^{\delta}}{\partial \delta_i} dx$ computed at $\delta = 0$.*

Proposition 4.4. *Under assumptions ($\mathcal{H}$), ($\mathcal{D}$), we have for $y \leq 0$ the decomposition*

$$G = E + S + R, \tag{4.11}$$

where

$$\begin{aligned} E(x, t; y) := \sum_{i=1}^{\ell} \sum_{a_k^- > 0} [c_{k,-}^{0,i}] \frac{\partial \bar{u}^{\delta}}{\partial \delta_i}(x) l_k^{-t} \left(errfn \left(\frac{y + a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right. \\ \left. - errfn \left(\frac{y - a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right), \end{aligned} \tag{4.12}$$

(4.13)

$$S(x,t;y) := \chi_{\{t\ge 1\}} \sum_{a_k^-<0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t}$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^->0} r_k^- l_k^{-t} (4\pi\beta_k^- t)^{-1/2} e^{-(x-y-a_k^- t)^2/4\beta_k^- t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right)$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^->0,\, a_j^-<0} [c_{k,-}^{j,-}] r_j^- l_k^{-t} (4\pi\bar\beta_{jk}^- t)^{-1/2} e^{-(x-z_{jk}^-)^2/4\bar\beta_{jk}^- t} \left(\frac{e^{-x}}{e^x+e^{-x}}\right),$$

$$+ \chi_{\{t\ge 1\}} \sum_{a_k^->0,\, a_j^+>0} [c_{k,-}^{j,+}] r_j^+ l_k^{-t} (4\pi\bar\beta_{jk}^+ t)^{-1/2} e^{-(x-z_{jk}^+)^2/4\bar\beta_{jk}^+ t} \left(\frac{e^{x}}{e^x+e^{-x}}\right),$$

with

$$z_{jk}^{\pm}(y,t) := a_j^{\pm}\left(t - \frac{|y|}{|a_k^-|}\right) \tag{4.14}$$

and

$$\bar\beta_{jk}^{\pm}(x,t;y) := \frac{x^{\pm}}{a_j^{\pm} t}\beta_j^{\pm} + \frac{|y|}{|a_k^- t|}\left(\frac{a_j^{\pm}}{a_k^-}\right)^2 \beta_k^-, \tag{4.15}$$

and

(4.16)

$$\begin{aligned} R(x,t;y) =& \\ &\mathbf{O}(e^{-\eta(|x-y|+t)}) \\ &+ \sum_k \mathbf{O}\left((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} \\ &+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge|y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2} e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}, \\ &+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge|y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}, \end{aligned}$$

for some η, $M>0$, where $x^{\pm}$ denotes the positive/negative part of x, indicator function $\chi_{\{|a_k^- t|\ge|y|\}}$ is one for $|a_k^- t|\ge|y|$ and zero otherwise, indicator function $\chi_{\{t\ge 1\}}$ is one for $t\ge 1$ and zero otherwise, and scattering coeffi-

cients $[c^{0,i}_{k,-}]$, $[c^{j,\pm}_{k,-}]$ *are constant, with*

$$\sum_{a_j^-<0}[c^{j,-}_{k,-}]r_j^- + \sum_{a_j^+>0}[c^{j,+}_{k,-}]r_j^+ + \sum_{i=1}^{\ell}[c^{0,i}_{k,-}]\int_{-\infty}^{+\infty}\frac{\partial \bar u^\delta}{\partial \delta_i}dx = r_k^- \tag{4.17}$$

for each k *(note: uniquely determined, by condition (D2)), and*

(4.18)
$$\begin{aligned}\sum_{a_k^->0}[c^{0,i}_{k,-}]l_k^- &= \sum_{a_k^+<0}[c^{0,i}_{k,+}]l_k^+ \\ &= \pi_i := (r_1^-,\dots,r_{p-1}^-,r_{p+l}^+,\dots,r_n^+,\int_{-\infty}^{+\infty}\frac{\partial \bar u^\delta}{\partial \delta_i}dx)^{-1}e_{n-i+1},\end{aligned}$$

where e_j *denotes the* j*th standard basis element, and with* $\frac{\partial \bar u^\delta}{\partial \delta_i}$ *always computed at* $\delta=0$. *Likewise, we have the derivative bounds*

(4.19)
$$\begin{aligned}&|R_x| = \\ &\mathbf{O}(e^{-\eta(|x-y|+t)}) \\ &+\sum_k \mathbf{O}\left((t+1)^{-1/2}t^{-1/2}e^{-\eta x^+}+e^{-\eta|x|}\right)t^{-1/2}e^{-(x-y-a_k^-t)^2/Mt} \\ &+\sum_{a_k^->0,\,a_j^-<0}\chi_{\{|a_k^-t|\ge|y|\}}\mathbf{O}((t+1)^{-1}t^{-1/2}e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}, \\ &+\sum_{a_k^->0,\,a_j^+>0}\chi_{\{|a_k^-t|\ge|y|\}}\mathbf{O}((t+1)^{-1}t^{-1/2}e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-},\end{aligned}$$

(4.20)
$$\begin{aligned}&|R_y| = \\ &\mathbf{O}(e^{-\eta(|x-y|+t)}) \\ &+\sum_k \mathbf{O}\left((t+1)^{-1/2}e^{-\eta x^+}+e^{-\eta|x|}\right)t^{-1}e^{-(x-y-a_k^-t)^2/Mt} \\ &+\sum_{a_k^->0,\,a_j^-<0}\chi_{\{|a_k^-t|\ge|y|\}}\mathbf{O}((t+1)^{-1}t^{-1/2}e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}e^{-\eta x^+}. \\ &+\sum_{a_k^->0,\,a_j^+>0}\chi_{\{|a_k^-t|\ge|y|\}}\mathbf{O}((t+1)^{-1}t^{-1/2}e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt}e^{-\eta x^-}.\end{aligned}$$

A symmetric decomposition holds for $y\ge 0$. *Moreover, for* $|x-y|/t$ *suffi-*

ciently large,

$$|G| \le Ce^{-\frac{|x-y|^2}{Mt}}. \tag{4.21}$$

Remark 4.5. Though it was not remarked in [**MaZ.3**], the terms E and S are continuous at $y=0$, a consequence of the respective scattering relations (4.18) and (4.17). (Note that values at $y=0$ correspond to time-asymptotic states described by the scattering relations, which depend only on mass and not position of data.)

Remark 4.6. The term $e^{-\eta(|x-y|+t)}$ in R and its derivatives corrects a minor omission in [**Z1**]. This term comes from the fact that, in the far field, E and S decay at this rate while entire G decays at faster Gaussian rate. The Gaussian decay (4.21) was proved but not stated in [**MaZ.3**]. The bound for R_x is given here only for the sake of completeness, and is not going to play a role in our calculations.

Remark 4.7. In [**Z1**] and [**MaZ.3**] the above bounds have been explicitly stated and proved for Lax case, and only some hints are given as about the overcompressive case. The proof for the overcompressive case, however, is not very different and can be achieved following the same outline given for Lax case.

Define $e_i, i=1,\cdots,\ell$ for $y<0$

$$\begin{aligned} e_i(y,t) := \sum_{a_k^->0} [c_{k,-}^{0,i}] l_k^{-t} \Bigg(& errfn\left(\frac{y+a_k^- t}{\sqrt{4\beta_k^- t}}\right) \\ & - errfn\left(\frac{y-a_k^- t}{\sqrt{4\beta_k^- t}}\right)\Bigg), \end{aligned} \tag{4.22}$$

Hence

$$E(x,t:y) = \sum_{i=1}^{\ell} \frac{\partial \bar{u}^\delta}{\partial \delta_i}(x) e_i(y,t) \tag{4.23}$$

and symmetrically for $y>0$. Define also

$$\tilde{G} = S + R. \tag{4.24}$$

We have the following bounds for $\tilde{G}$ and e_i's:

Lemma 4.8. *Under assumptions* $(\mathcal{H})$ *and* $(\mathcal{D})$ *there holds*

$$|\int_{-\infty}^{+\infty} \tilde{G}(\cdot,t;y)f(y)dy|_{L^p} \le C\min\{|f|_{L^p}, t^{-\frac{1}{2}(1-1/p)}|f|_{L^1}\}, \tag{4.25}$$

$$|\int_{-\infty}^{+\infty} \tilde{G}_y(\cdot,t;y)f(y)dy|_{L^p} \le C\min\{t^{-1/2}|f|_{L^p}, t^{-\frac{1}{2}(1-1/p)-1/2}|f|_{L^1}\}, \tag{4.26}$$

for all $t \ge 0$*,* $f \in L^1 \cap L^p$*, some* $C > 0$*.*

Lemma 4.9. *The kernels* e_i*'s satisfy*

$$|e_{i_y}(\cdot,t)|_{L^p}, |e_{i_t}(\cdot,t)|_{L^p} \le Ct^{-\frac{1}{2}(1-1/p)}, \tag{4.27}$$

$$|e_{i_{ty}}(\cdot,t)|_{L^p} \le Ct^{-\frac{1}{2}(1-1/p)-1/2}, \tag{4.28}$$

for all $t > 0$*. Moreover, for* $y \le 0$ *we have the pointwise bounds*

$$|e_{i_y}(y,t)|, |e_{i_t}(y,t)| \le C\sum_{a_k^- > 0} t^{-\frac{1}{2}} e^{-\frac{(y+a_k^- t)^2}{Mt}},$$

$$|e_{i_{ty}}(y,t)| \le C\sum_{a_k^- > 0} t^{-1} e^{-\frac{(y+a_k^- t)^2}{Mt}},$$

for $M > 0$ *sufficiently large (i.e.,* $> 4b_\pm$*), and symmetrically for* $y \ge 0$*.*

Proof. See **[Z1]** and **[MaZ.4]**. □

Let $\tilde{u}$ solve (4.1), and, using (D2), assume that

$$\int_{-\infty}^{+\infty} \tilde{u}(x,0) - \bar{u}(x) = \sum_{a_j^- < 0} m_j r_j^- + \sum_{a_j^+ > 0} m_j r_j^+ + \sum_{i=1}^{\ell} \int c_i \frac{\partial \bar{u}^\delta}{\partial \delta_i}$$

with m_i's and c_i's small enough. Using the Implicit Function Theorem, we can find δ_0 such that

$$\int_{-\infty}^{+\infty} \tilde{u}(x,0) - \bar{u}^{\delta_0}(x) = \sum_{a_j^- < 0} m_j' r_j^- + \sum_{a_j^+ > 0} m_j' r_j^+$$

where each m_i' is just "slightly" different from m_i. Notice that this way we have no "mass" in any $\int \frac{\partial \bar{u}^\delta}{\partial \delta_i}$ direction anymore. Therefore, by replacing $\bar{u}$ with $\bar{u}^{\delta_0}$ and without loss of generality, we can assume $\delta_0 = 0$ and

$$\int_{-\infty}^{+\infty} \tilde{u}(x,0) - \bar{u}(x) = \sum_{a_j^- < 0} m_j r_j^- + \sum_{a_j^+ > 0} m_j r_j^+.$$

Remark 4.10. In Lax case shock waves, $\bar{u}^\delta(x) = \bar{u}(x+\delta)$, hence $\frac{\partial \bar{u}^\delta}{\partial \delta} = u'(x)$, and δ_0 can be explicitly computed: $\delta_0 = c_1$.

Let $u(x,t) = \tilde{u}(x,t) - \bar{u}(x)$ and use Taylor's expansion around $\bar{u}^{\delta(t)}(x)$ to find

$$u_t + (A(x)u)_x - (B(x)u_x)_x = -(\Gamma(x)(u,u))_x + Q(u,u_x)_x, \tag{4.29}$$

where $\Gamma(x)(u,u) = d^2 f(\bar{u})(u,u) - d^2 B(\bar{u})(u,u)\bar{u}_x$ and

$$Q(u,u_x) = \mathbf{O}(|u||u_x| + |u|^3).$$

Denote $\Gamma^\pm = \Gamma(\pm\infty)$, and note that we have similar statements to (4.8) for $\Gamma(x) - \Gamma^\pm$. Define constant coefficients $b_{ij}^\pm$ and $\Gamma_{ijk}^\pm$ to satisfy

$$\Gamma^\pm(r_j^\pm, r_k^\pm) = \sum_{i=1}^n \Gamma_{ijk}^\pm r_i^\pm, \quad B^\pm r_j^\pm = \sum_{i=1}^n b_{ij}^\pm r_i^\pm \tag{4.30}$$

hence of course $\beta_i^\pm = b_{ii}^\pm$, and denote $\gamma_i^\pm := \Gamma_{iii}^\pm$.

Remark 4.11. As it is pointed out in [**Liu1, Liu2, Liu3**], $\gamma_i^\pm \neq 0$ is equivalent to *genuine nonlinearity* of the i^{th} field, and $\gamma_i^\pm = 0$ means that the i^{th} field is *linearly degenerate*.

We define diffusion waves along outgoing modes: for $a_i^- < 0$ define the diffusion wave φ_i to solve:

$$\begin{cases} \varphi_t^i + a_i^- \varphi_x^i - \beta_i^- \varphi_{xx}^i = -\gamma_i^-({\varphi^i}^2)_x & \text{for} \quad t > -1 \\ \varphi^i(x,-1) = m_i \delta_0 & \qquad\quad t = -1 \end{cases} \tag{4.31}$$

and likewise for $a_i^+ > 0$, define φ_i to be the solution of

$$\begin{cases} \varphi_t^i + a_i^+ \varphi_x^i - \beta_i^+ \varphi_{xx}^i = -\gamma_i^+({\varphi^i}^2)_x & \text{for} \quad t > -1 \\ \varphi^i(x,-1) = m_i \delta_0 & \qquad\quad t = -1 \end{cases} \tag{4.32}$$

and set

$$\varphi = \sum_{a_i^-<0} \varphi^i r_i^- + \sum_{a_i^+>0}^{n} \varphi^i r_i^+.$$

Let $v := u - \varphi - \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$, where $\delta(t) = (\delta_1(t), \cdots, \delta_\ell(t))^{\mathrm{tr}}$ is to be defined later, assuming $\delta(0) = 0$. Notice that

$$\int_{-\infty}^{+\infty} v(x,0)dx = 0, \tag{4.33}$$

so if $V_0 = \int_{-\infty}^{x} v(y,0)dy$ then by lemma 3.1 $V_0 \in L^1$ and $|V_0|_{L^1} \leq |xv_0|_{L^1}$.

Replacing u with $v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t)$ in (4.29) ($\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ computed at $\delta = \delta_0 = 0$), and using the fact that $\frac{\partial \bar{u}^\delta}{\partial \delta_i}$ satisfies the linear time independent equation $Lv = 0$, we will have

$$v_t - Lv = \Psi(x,t) + \mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t))_x + \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta}(t), \tag{4.34}$$

where

(4.35)

$$\begin{aligned}\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) = \mathbf{O}(&|v|^2 + |\varphi||v| + |v||\frac{\partial \bar{u}^\delta}{\partial \delta}\delta| + |\varphi||\frac{\partial \bar{u}^\delta}{\partial \delta}\delta| + |\frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^2 \\ &+ |(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_x| + |\varphi + v + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^3).\end{aligned}$$

and $\Psi := -\varphi_t - (A(x)\varphi)_x + (B(x)\varphi_x)_x - (\Gamma(x)(\varphi,\varphi))_x$. For Ψ we write

(4.36)

$$\begin{aligned}\Psi(x,t) =& - (\varphi_t + A\varphi_x - B\varphi_{xx} + \Gamma(\varphi,\varphi)_x) \\ =& - \sum_{a_i^-<0} \varphi_t^i r_i^- + (A(x)\varphi^i r_i^-)_x - (B(x)\varphi_x^i r_i^-)_x + (\Gamma(x)(\varphi^i r_i^-, \varphi^i r_i^-))_x \\ &- \sum_{a_i^+>0} \varphi_t^i r_i^+ + (A(x)\varphi^i r_i^+)_x - (B(x)\varphi_x^i r_i^+)_x + (\Gamma(x)(\varphi^i r_i^+, \varphi^i r_i^+))_x \\ &- \sum_{i\neq j} (\varphi_i \varphi_j \Gamma(x)(r_i^\pm, r_j^\pm))_x.\end{aligned}$$

Let us write a typical term of the first summation ($a_i^- < 0$)in the following

form:

(4.37)

$$\begin{aligned}
&\varphi^i_t r^-_i + (A(x)\varphi^i r^-_i)_x - (B(x)\varphi^i_x r^-_i)_x + (\Gamma(x)(\varphi^i r^-_i, \varphi^i r^-_i))_x \\
&\quad = \big[(A(x) - A^-)\varphi^i r^-_i - (B(x) - B^-)\varphi^i_x r^-_i + (\Gamma(x) - \Gamma^-)(\varphi^i r^-_i, \varphi^i r^-_i)\big]_x \\
&\quad + \varphi^i_t r^-_i + (\varphi^i_x A^- r^-_i) - (\varphi^i_{xx} B^- r^-_i) + ((\varphi^i)_x \Gamma^-(r^-_i, r^-_i)).
\end{aligned}$$

Now we use the definition of φ^i in (4.31) and the definition of coefficients b_{ij} and Γ_{ijk} in (4.30) to write the last part of (4.37) in the following form:

$$\begin{aligned}
\varphi^i_t r^-_i + (\varphi^i_x A^- r^-_i) &- (\varphi^i_{xx} B^- r^-_i) + ((\varphi^i)_x \Gamma^-(r^-_i, r^-_i)) \\
&= -\varphi^i_{xx} \sum_{j \neq i} b^-_{ij} r^-_j - (\varphi^i)^2_x \sum_{j \neq i} \Gamma^-_{jii} r^-_j .
\end{aligned} \tag{4.38}$$

Similar statements hold for $a^+_i > 0$ with minus signs replaced with plus signs.

Later we will need some estimates for v_x. Short time estimates gives us the necessary bounds. Let us first provide the requisite short time existence/regularity theory for general quasilinear parabolic systems, using the paramatrix method of Levi [**LSU, Le**].

Proposition 4.12. *Let $\hat{A}(x,t)$, $\hat{B}(x,t)$, and $\hat{C}(x,t)$ be uniformly bounded in L^∞ and $C^{(0,0)+(\alpha,\alpha/2)}(x,t)$, $0 < \alpha, < 1$, taking values on a compact set, with $Re\,\sigma(\hat{B})$ positive and bounded strictly away from zero. Then, for $0 < t < T$, T sufficiently small, there is a Green's function $\hat{G}(x,t;y,s) \in C^{1,0}(x,t)$ associated with the Cauchy problem for*

$$v_t = \hat{C}v + (\hat{A}v)_x + (\hat{B}v_x)_x, \quad v \in \mathbb{R}^n, \tag{4.39}$$

satisfying bounds

$$|D^j_x \hat{G}(x,t;y,s)| \le Ct^{-(j+1)/2} e^{-(x-y)^2/M(t-s)}, \quad j = 0, 1, \tag{4.40}$$

where C, M, $T > 0$ depend only on the bounds on the coefficients and on the lower bound on $Re\,\sigma(\hat{B})$.

Proof. See [**ZH, LSU**]. □

The following lemma provides us with the short time estimates we need for v_x.

Lemma 4.13. *Given the above setting, and assuming v remains bounded for all the time, we will have:*

$$|v_x(\cdot,t)|_{L^p} \le \begin{cases} C(|v(\cdot,t-1)|_{L^p} + |\mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)|_{L^p}), & \text{for } t \ge 1 \\ C(t^{-\frac{1}{2}}|v_0|_{L^p}| + |\mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)|_{L^p}), & \text{for } t \le 1. \end{cases} \tag{4.41}$$

where

$$\mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) = -\frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta} + \mathbf{O}(|\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta||\varphi_x + (\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_x| + e^{-k|x|}|\varphi|). \tag{4.42}$$

for some $k > 0$

Proof. $\tilde{u}_t + f(\tilde{u})_x = (B(\tilde{u})\tilde{u}_x)_x$ and $f(\bar{u})_x = (B(\bar{u})\bar{u}_x)_x$ implies

(4.43)

$$\begin{aligned} &v_t + \varphi_t + \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta} + f(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta + v)_x - f(\bar{u})_x \\ &\qquad = (B(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta + v)(\bar{u}_x + \varphi_x + (\frac{\partial \bar{u}^\delta}{\partial \delta})_x\delta + v_x))_x - (B(\bar{u})\bar{u}_x)_x. \end{aligned}$$

Hence:

(4.44)

$$\begin{aligned} &v_t + (f(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta + v) - f(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta))_x \\ &-\Big((B(\bar{u}+\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta + v) - B(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta))(\bar{u}_x + \varphi_x + (\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_x)\Big)_x \\ &-(B(\bar{u}+\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta + v)v_x)_x \\ &= -\varphi_t - \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta} - (f(\bar{u} + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) - f(\bar{u}))_x \\ &+((B(\bar{u}+\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) - B(\bar{u}))\bar{u}_x)_x \\ &+(B(\bar{u}+\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\varphi_x + (\frac{\partial \bar{u}^\delta}{\partial \delta})_x\delta)_x \\ &=: \mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta). \end{aligned}$$

It is not difficult to observe that

(4.45)

$$\begin{aligned} \mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta) = -\varphi_t - (A(x)(\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta))_x + (B(x)(\varphi_x + \frac{\partial \bar{u}_x^\delta}{\partial \delta}\delta)_x - \frac{\partial \bar{u}^\delta}{\partial \delta}\dot{\delta} \\ +\mathbf{O}(|\varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta||\varphi_x + (\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_x|. \end{aligned}$$

Using (4.36), (4.37), (4.8) and the fact that $L\frac{\partial \bar{u}^\delta}{\partial \delta} = 0$, we conclude (4.42). We use

$$f(\eta + v) - f(\eta) = \int_0^1 df(\eta + \theta v) d\theta v$$

and similar equation for B in order to write (4.44) in the form

$$v_t + (\hat{A}(x,t)v)_x - (\hat{B}(x,t)v_x)_x = \mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta). \tag{4.46}$$

Now $\hat{A}$ and $\hat{B}$ depend on v. Momentarily assume v (hence $\hat{A}$ and $\hat{B}$) is in $C^{(0,0)+(\alpha,\alpha/2)}(x,t)$.

$$\begin{aligned} v_x(x,t) &= \int_{-\infty}^{+\infty} \hat{G}_x(x,t;y,t-T) v(y,t-T)\, dy \\ &\quad + \int_{t-T}^{t} \int_{-\infty}^{+\infty} \hat{G}_x(x,t;y,s) \mathcal{M}(y,s)\, dy\, ds. \end{aligned} \tag{4.47}$$

By Duhamel's principle, where $\hat{G}$ is the Green's function for (4.46), and using the $\hat{G}_x$ bounds of proposition 4.12 for divergence-form operators, we find that

$$|v_x(\cdot,t)|_\infty \le C(T)^{-1/2} |v(\cdot, t-T)|_\infty + C(T)^{1/2} |\mathcal{M}|_\infty.$$

In particular, for $t \ge T$, we obtain a uniform Hölder (indeed, Lipshitz) bound on $v(\cdot,t)$ depending only on the L^∞ norm of $v(\cdot, t-T)$. By the (standard) method of extension, we thus obtain uniform Hölder continuity of v so long as $|v|$ remains bounded. (4.41) follows using (4.47) and taking (without loss of generality) $T = 1$. For a more detailed discussion see **[ZH]** (section 11). □

Now we employ Duhamel's principle to get from (4.34) :

$$
\begin{aligned}
v(x,t) &= \int_{-\infty}^{+\infty} G(x,t;y)v_0(y)dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(s))_y(y,s)dy\,ds, \\
&+ \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\Psi(y,s)dy\,ds \\
&+ \delta(t)\cdot \frac{\partial \bar{u}^\delta}{\partial \delta} \\
&= -\int_{-\infty}^{+\infty} G_y(x,t;y)V_0(y)dy \\
&- \int_0^t \int_{-\infty}^{+\infty} G_y(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(s))(y,s)dy\,ds, \\
&+ \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\Psi(y,s)dy\,ds \\
&+ \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(t).
\end{aligned}
\tag{4.48}
$$

(the last part of the above equation follows from $\int_{-\infty}^{+\infty} G(x,t;y)\frac{\partial \bar{u}^\delta}{\partial \delta_i}(y)dy = e^{Lt}\frac{\partial \bar{u}^\delta}{\partial \delta_i} = \frac{\partial \bar{u}^\delta}{\partial \delta_i}$ and $\delta(0)=0$). Set

$$
\begin{aligned}
\delta_i(t) &= \int_{-\infty}^{\infty} e_{i_y}(y,t)V_0(y)dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} e_{i_y}(y,t-s)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta(s))dyds \\
&- \int_0^t \int_{-\infty}^{+\infty} e_i(x,t-s;y)\Psi(y,s)dy\,ds.
\end{aligned}
\tag{4.49}
$$

Using (4.48), (4.49) and $G = E + \tilde{G}$ we obtain:

$$
\begin{aligned}
v(x,t) &= \int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y)V_0(y)dy \\
&- \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\mathcal{F}(\varphi, v, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds, \\
&+ \int_0^t \int_{-\infty}^{+\infty} \tilde{G}(x,t-s;y)\Psi(y,s)dy\,ds.
\end{aligned}
\tag{4.50}
$$

We are now in possession of the necessary tools to state the following theorem:

Theorem 4.14. *Let $(\mathcal{H})$ and $(\mathcal{D})$ hold, and $|u_0|_{L^1}$, $|xu_0|_{L^1}$, $|u_0|_{L^\infty} \leq E_0$, E_0 sufficiently small (these assumption on u_0 are being inherited by v_0). Assume the above setting and $u = v + \varphi + \frac{\partial \bar{u}^\delta}{\partial \delta}\delta$, then for any ϵ, $0 < \epsilon < \frac{1}{8}$,*

$$|v(\cdot,t)|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}, \tag{4.51}$$

$$|\delta(t)| \leq CE_0(1+t)^{-\frac{1}{2}+\epsilon}, \tag{4.52}$$

$$|\dot{\delta}(t)| \leq CE_0(1+t)^{-1+\epsilon}, \tag{4.53}$$

for any p, $1 \leq p \leq \infty$, and with C independent of p (but depending on ϵ).

An immediate consequence to this theorem is the following corollary, which is almost (up to an ϵ) Liu's result.

Corollary 4.15.

$$|\tilde{u} - \bar{u}^{\delta_0} - \varphi|_{L^p} \leq \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}} & \text{for } 1 \leq p \leq \frac{2}{1+8\epsilon}, \\ (1+t)^{-\frac{1}{2}+\epsilon} & \text{for } p \geq \frac{2}{1+8\epsilon}. \end{cases} \tag{4.54}$$

However, our approach yields more information about the behavior of the perturbation, as we can track the shock location: by (1.3) and the comment right after, the following corollary follows.

Corollary 4.16.

$$|\tilde{u} - \bar{u}^{\delta_0+\delta(t)} - \varphi|_{L^p} \leq (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}} \tag{4.55}$$

for all p.

Proof of Theorem 4.14. Fixing ϵ, define

(4.56)

$$\zeta(t) := \sup_{0\leq s\leq t, 1\leq p\leq\infty} |u(\cdot,s)|_{L^p}(1+s)^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{4}} + \sup_{0\leq s\leq t}|\delta(s)|(1+s)^{\frac{1}{2}-\epsilon} + \sup_{0\leq s\leq t}|\dot{\delta}(s)|(1+s)^{1-\epsilon}.$$

Our aim is to show that

$$\zeta(t) \le C(E_0 + \zeta^2(t)) \tag{4.57}$$

and then use a straightforward continuous induction. Equivalent to (4.57) is

$$|v(\cdot, s)|_{L^p} \le C(E_0 + \zeta^2(t))(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}} \tag{4.58}$$

and

$$|\delta(s)| \le C(E_0 + \zeta^2(t))(1+s)^{-\frac{1}{2}+\epsilon} \tag{4.59}$$

and a similar statement for $\dot{\delta}$. We need to take three steps:

step 1: bounds for $|\mathbf{v}|_{\mathbf{L^p}}$: By corollary 4.2, we have $|\frac{\partial \bar{u}^\delta}{\partial \delta_i}| \sim e^{-k|x|}$ and $|(\frac{\partial \bar{u}^\delta}{\partial \delta_i})_x| \sim e^{-k|x|}$ for some $k > 0$, hence, for $0 \le s \le t$ we have:

$$|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta(s)|_{L^p} \le C\zeta(t)(1+s)^{-\frac{1}{2}+\epsilon}, \tag{4.60}$$

$$|(\frac{\partial \bar{u}^\delta}{\partial \delta})_x\delta(s)|_{L^p} \le C\zeta(t)(1+s)^{-\frac{1}{2}+\epsilon}, \tag{4.61}$$

$$|\varphi(\cdot, s)|_{L^p} \le CE_0(1+s)^{-\frac{1}{2}(1-\frac{1}{p})}, \tag{4.62}$$

$$|\varphi_x(\cdot, s)|_{L^p} \le CE_0(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}. \tag{4.63}$$

Lemma 4.13 provides us with the necessary bounds for v_x. Note that by lemma 2.10 and the above bounds, we have the following bounds for $\mathcal{M}$ in (4.42):

$$|\mathcal{M}(\varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\cdot, s)|_{L^p} \le C(E_0 + \zeta(t))(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}} \tag{4.64}$$

when $1 \le p \le 2$. Therefore,

$$|v_x(\cdot, s)|_{L^p} \le \begin{cases} C(E_0 + \zeta(t))s^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}, & \text{for } s \ge 1 \\ C(E_0 + \zeta(t))s^{-\frac{1}{2}}, & \text{for } s \le 1. \end{cases} \tag{4.65}$$

Now let $p^* = \frac{2}{1+8\epsilon}$, hence $\frac{1}{2}(1 - \frac{1}{p^*}) + \frac{3}{4} = 1 - 2\epsilon$. With bounds for φ, δ, v and v_x we obtain:

$$|\mathcal{F}(v, \varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\cdot, s)|_{L^p} \le C(E_0 + \zeta(t)^2)s^{-\frac{1}{2}(1-1/p)-\frac{3}{4}} \tag{4.66}$$

whenever $1 \le p \le p^*$.

When $t \le 1$, then

$$
\begin{aligned}
&|v(\cdot,t)|_{L^p} \le C|v_0|_{L^p} \\
&+ \int_0^t |\tilde{G}_y|_{L^1} |\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)|_{L^p}(s)ds \\
&+ \int_0^t (|\tilde{G}|_{L^1}||\psi(y,s)|_{L^p}(s)ds \\
&\le CE_0 + (E_0+\zeta(t)^2)\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}ds + CE_0\int_0^t (1+s)^{\frac{1}{2}} \\
&\le C(E_0+\zeta(t)^2) \le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}.
\end{aligned}
\tag{4.67}
$$

For $t \ge 1$ we use again Haussdorf-Young inequality to obtain:

$$
\begin{aligned}
&|\int_{-\infty}^{+\infty} \tilde{G}_y(x,t;y)V_0(y)dy|_{L^P} \\
&\le |V_0|_{L^1}|\tilde{G}_y|_{L^p} \le CE_0 t^{-\frac{1}{2}(1-1/p)-\frac{1}{2}},
\end{aligned}
\tag{4.68}
$$

and

$$
\begin{aligned}
&|\int_0^{t/2}\int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds|_{L^p} \\
&\qquad \le \int_0^{t/2} |\tilde{G}_y|_{L^p}|\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\cdot,s)|_{L^1}ds \\
&\qquad \le C(E_0+\zeta(t)^2)\int_0^{t/2}(t-s)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}s^{-\frac{3}{4}}ds \\
&\qquad \le C(\zeta_0+\zeta(t)^2)t^{-\frac{1}{2}(1-1/p)-\frac{1}{4}} \\
&\qquad \le 2C(\zeta_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}.
\end{aligned}
\tag{4.69}
$$

If $1 \le p \le p^*$, then

$$
\begin{aligned}
&|\int_{t/2}^{t}\int_{-\infty}^{+\infty} \tilde{G}_y(x,t-s;y)\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds|_{L^p} \\
&\qquad \le \int_{t/2}^{t} |\tilde{G}_y|_{L^1}|\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\cdot,s)|_{L^p}ds \\
&\qquad \le C(E_0+\zeta(t)^2)\int_{t/2}^{t}(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}(1-1/p)-\frac{3}{4}}ds \\
&\qquad \le C(E_0+\zeta(t)^2)t^{-\frac{1}{2}(1-1/p)-\frac{1}{4}} \\
&\qquad \le 2C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}.
\end{aligned}
\tag{4.70}
$$

If $p \geq p^*$, then choose q so that $\frac{1}{p}+1=\frac{1}{p^*}+\frac{1}{q}$, and then

$$
\begin{aligned}
|\int_{t/2}^{t}\int_{-\infty}^{+\infty} & \tilde{G}_y(x,t-s;y)\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(y,s)dy\,ds|_{L^p} \\
& \leq \int_{t/2}^{t} |\tilde{G}_y|_{L^q}|\mathcal{F}(v,\varphi,\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)(\cdot,s)|_{L^{p^*}}ds \\
& \leq C(E_0+\zeta(t)^2)\int_{t/2}^{t}(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}s^{-\frac{1}{2}(1-\frac{1}{p^*})-\frac{3}{4}}ds \\
& \leq C(\zeta_0+\zeta(t)^2)t^{-\frac{1}{2}(1-\frac{1}{p^*})-\frac{3}{4}+\frac{1}{2q}} \\
& = C(E_0+\zeta(t)^2)t^{-\frac{1}{2}(1-1/p)-\frac{1}{4}} \\
& \leq 2C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}}.
\end{aligned}
\tag{4.71}
$$

It remains to show

$$
|\int_0^t\int_{-\infty}^{+\infty}\tilde{G}(x,t-s;y)\Psi(y,s)dyds|_{L^p} \leq E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}.
$$

By (4.36),(4.37) and (4.38) we have to estimate the following: First, the terms in the form $(\varphi_i\varphi_j\Gamma(x)(r_i^{\pm},r_j^{\pm}))_x$, in (4.36). By lemma 2.9, $(\varphi_i\varphi_j\Gamma(x)(r_i^{\pm},r_j^{\pm}))$ is of order $E_0^2\mathbf{O}(e^{-\eta t})$,, so we can use similar calculations as before to conclude that $|\int_0^t\int_{-\infty}^{+\infty}\tilde{G}_y(x,t-s;y)(\varphi_i\varphi_j\Gamma(x)(r_i^{\pm},r_j^{\pm}))dy\,ds|_{L^p} = \mathbf{O}(E_0(1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{4}})$

Next, the terms in the second line of (4.37), i.e., in the form $((A(x)-A^-)\varphi^i r_i^-)_x$ or similar forms, of which lemma 2.10 together with (4.8) and similar bounds for Γ take care.

Finally, the terms in the form $\varphi^i_{xx}b^-_{ij}r^-_j$ and $(\varphi^i)^2_x\Gamma^-_{jii}r^-_j$, $i\neq j$, in (4.38). The L_p norm of these terms is of order $(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$, so not decaying fast enough to use calculations similar to what we have already done, so we have to use the results in section 2. Assume $a_i^-<0$; we examine the integration of $(\varphi^i)^2_x r^-_j$ against the different terms of $S(x,t-s;y)$. As $l_k^{-tr}r_j^-=0$ if $k\neq j$ the terms of concern in S integrated against $(\varphi^i)^2_{xx}r^-_j$ are as following: For $y<0$, the terms in the first line of (4.13), gives us:

$$
\int_0^t\int(4\pi\beta_j^-(t-s))^{-1/2}e^{-(x-y-a_j^-(t-s))^2/4\beta_j^-(t-s)}(\varphi_i)^2_y dy\,ds,
\tag{4.72}
$$

which, by (2.6) and (2.7), is of order $E_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}}$ (as $i\neq j$). The second line of (4.13) does not comprise anything. The third line of S comprises:

(4.73)
$$\int_0^t \int \chi_{\{t-s\geq 1\}}[c^{j,-}_{k,-}](4\pi\bar\beta^-_{jk}(t-s))^{-1/2}e^{-(x-z^-_{jk})^2/4\bar\beta^-_{jk}(t-s)}\left(\frac{e^{-x}}{e^x+e^{-x}}\right)(\varphi_i)^2_y dy\, ds$$

with $a^-_k>0, a^-_j<0$, and z^-_{jk} and $\bar\beta^-_{jk}$ computed at $t-s$. Note in this case the convection and diffusion coefficients are not constant. To make a brief presentation of the relevant calculations, we first notice that the biggest part in (4.73) is in the cone $a^-_j(t-s)/2 \leq x - z^-_{jk} \leq -a^-_j(t-s)/2$, or equivalently, $3a^-_j(t-s)/2 \leq x - a^-_j y/a^-_k \leq a^-_j(t-s)/2$ (outside this cone we have a negligible term). On this interval the diffusion coefficient $\bar\beta^-_{jk}$ can be bounded from above by a constant β^*. The derivatives of $\bar\beta^-_{jk}(t-s)$ also can be bounded from above similarly. As a consequence the y and t derivatives of this part of S satisfy the bounds used in proposition 2.1 in the cone just mentioned. Also we make a change of coordinates $z=\frac{a^-_j y}{a^-_k}$ to see that , this part of (4.73) can be estimated the same way one would estimate

$$\int\int g(x-z-a^-_j(t-s),\beta^*(t-s))g(z-a^-_i a^-_j s/a^-_k, s)^2_y dyds.$$

using the same process as in proposition 2.1 and subsequent results. Notice that $a^-_i a^-_j/a^-_k \neq a^-_j$, i.e., the different speed of the Gaussian kernel in the Green function and the diffusion wave.

For $y>0$, the corresponding terms in second line of S in (4.13) gives:

$$\int_0^t \int (4\pi\beta^+_k(t-s))^{-1/2}e^{-(x-y-a^+_k(t-s))^2/4\beta^+_k(t-s)}\left(\frac{e^{x}}{e^x+e^{-x}}\right)(\varphi_i)^2_y dy\, ds.$$

The case would be different from (4.72) if $a^+_k = a^-_i$. In this case we have a term like:

$$\left(\frac{e^{x}}{e^x+e^{-x}}\right)\int_0^t\int g(x-y-a(t-s),t-s)g^2(y-as,s)dy\,ds, \tag{4.74}$$

with $a<0$, which, by some elementary calculations, is less than or equal to:

$$\frac{Ce^x}{e^x+e^{-x}}g(x-at,Mt). \tag{4.75}$$

Now use lemma 2.10.
The terms in the remainder $R(x,t;y)$ can be dealt with in a similar way.

step 2: bounds for $\delta(\mathbf{t})$: In order to show (5.27) holds we investigate the integration of e_{i_y} and e_i against each term in $V_0, \mathcal{F}(v, \varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)$ and Ψ, respectively. To that end, we will have the following (assume $t \geq 1$).

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e_{i_y}(y,t) V_0(y) dy \\
&\quad \leq |e_{i_y}|_{L^\infty} |V_0|_{L^1} dy \\
&\quad \leq C t^{-\frac{1}{2}},
\end{aligned} \tag{4.76}$$

and

$$\begin{aligned}
&|\int_0^t \int_{-\infty}^{+\infty} e_{i_y}(y, t-s) \mathbf{O}(|v|^2)(y,s) dy\, ds| \\
&\quad \leq C \int_0^{t/2} |e_{i_y}|_{L^\infty}(y, t-s) ||v|^2|_{L^1}(y,s) ds \\
&\qquad + C \int_{t/2}^t |e_{i_y}|_{L^1}(y, t-s) ||v|^2|_{L^\infty}(y,s) ds \\
&\quad \leq C\zeta^2(t) (\int_0^{t/2} (t-s)^{-\frac{1}{2}} (1+s)^{-1}(y,s) ds + \int_{t/2}^t (1+s)^{-\frac{3}{2}} ds) \\
&\quad \leq C\zeta^2(t)(1+t)^{-\frac{1}{2}+\epsilon}.
\end{aligned} \tag{4.77}$$

Similarly for $\mathbf{O}(|v||v_x|)$. For $\mathbf{O}((\frac{\partial \bar{u}^\delta}{\partial \delta}\delta)^2)$ we use lemma 2.12 to get

$$\begin{aligned}
&|\int_0^t \int_{-\infty}^{+\infty} e_{i_y}(y, t-s) \mathbf{O}(|\frac{\partial \bar{u}^\delta}{\partial \delta}\delta|^2)(y,s) dy\, ds \\
&\quad \leq C\zeta^2(t) \int_0^t (1+s)^{-\frac{1}{2}+\epsilon} \int_{-\infty}^{+\infty} e_{i_y}(y, t-s) (|\frac{\partial \bar{u}^\delta}{\partial \delta}|^2)(y,s) dy\, ds \\
&\quad \leq C\zeta^2(t) \int_0^t (1+s)^{-\frac{1}{2}+\epsilon} (t-s)^{-\frac{1}{2}} e^{-\eta(t-s)} ds \\
&\quad \leq C\zeta^2(t)(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{4.78}$$

For $\mathbf{O}(|\varphi|^2)$ we use the fact that both φ and e_{i_y} are the summation of signals like convecting heat kernels, moving away from shock. Hence using lemma 2.11 gives us:

$$|\int_0^t \int_{-\infty}^{+\infty} e_{i_y}(y, t-s) \mathbf{O}(|\varphi|^2)(y,s) dy\, ds \leq C E_0 (1+t)^{-\frac{1}{2}}. \tag{4.79}$$

All the other terms in $\mathcal{F}(v, \varphi, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)$ and Ψ can be treated with similar methods.

step 3: bounds for $\dot{\delta}(\mathbf{t})$: Very similar to the previous calculations for $\delta(t)$. $\square$

Remark 4.17. As for the conditions on initial data u_0, it is in fact enough to assume (as Liu and others have done) that $u_0 = \mathbf{O}(1+|x|)^{-\frac{3}{2}}$. To see briefly why this works notice that the only part we should change in our argument is the linear part (4.68). Now for this part, it is enough to consider the convolution of a Gaussian signal, say g(x,t), against v_0. As $|v_0| \sim (1+|x|)^{-\frac{3}{2}}$ and $|V_0| \sim (1+|x|)^{-\frac{1}{2}}$, we consider $\int_{-\infty}^{+\infty} g_y(x-y,t)V_0(y)dy$ when $|x| \leq \sqrt{t}$, and $\int_{-\infty}^{+\infty} g(x-y,t)v_0(y)dy$ when $|x| \leq \sqrt{t}$. It is not difficult to observe, using Howard's lemma 2.4, that

$$|\int_{-\infty}^{+\infty} \tilde{G}(x-y,t)v_0(y)dy| \leq C\big(\chi_{\{|x|\leq\sqrt{t}\}}t^{-\frac{1}{2}}(1+|x|)^{-\frac{1}{2}}+\chi_{\{|x|\geq\sqrt{t}\}}(1+|x|)^{-\frac{3}{2}}\big).$$

The necessary L^1 bounds then follows immediately. To obtain L^∞ bounds in the case $|x| \leq \sqrt{t}$ we use lemma 2.3, to see that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g_y(x-y,t)V_0(y)dy \\
&\qquad \leq Ct^{-\frac{1}{2}}\int_{-\infty}^{+\infty} t^{-\frac{1}{2}}e^{-(x-y)^2/4t}y^{-\frac{1}{2}}dy \\
&\qquad \leq Ct^{-\frac{3}{4}}\int_{-\infty}^{+\infty} e^{-y^2/8t}(\frac{y}{\sqrt{t}})^{-\frac{1}{2}}\frac{dy}{\sqrt{t}} \\
&\qquad \leq Ct^{-\frac{3}{4}}.
\end{aligned}$$

Other L^p bounds follow using interpolation.

Remark 4.18. It is an easy observation that, in order to have (4.66) for $1 \leq p \leq p^* < 2$, it is enough to have L^2 bounds for v_x in (4.65), as v_x is always multiplied by a favorable term in $\mathcal{F}$. This will become important when we consider the real viscosity case, as we have only good L^2 bounds (using energy estimates) for v_x in that case .

Remark 4.19. The Analysis can go through in the case $df(u_\pm)$ is not strictly hyperbolic, provided that $A_\pm$ and $B_\pm$ are simultaneously symmetrizable, by replacing Green function bounds with more general bounds given in proposition 5.10 of **[Z.3]**, and replacing the diffusion waves φ_i's with the "multi-mode diffusion waves" of **[Ch]** and **[LZe]**. The same remark is applicable in the real viscosity case.

5 Real viscosity case

In this section we follow closely the notations and assumptions used in [**Z.3**]. Consider a general system of *viscous conservation laws*

$$U_t + F(U)_x = (B(U)U_x)_x, \qquad x \in \mathbb{R}; U,\ F \in \mathbb{R}^n;\ B \in \mathbb{R}^{n\times n}, \tag{5.1}$$

modeling flow in a compressible medium. We assume

$$U = \begin{pmatrix} u^I \\ u^{II} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \tag{5.2}$$

$u^I \in \mathbb{R}^{n-r}$, $u^{II} \in \mathbb{R}^r$, and

$$Re\sigma b_2 \geq \theta, \tag{5.3}$$

with $\theta > 0$.

Again we consider the *viscous shock wave* solutions of (5.1), which are in the form:

$$U(x,t) = \bar{U}(x), \quad \lim_{x\to\pm\infty} \bar{U}(x) = U_\pm, \tag{5.4}$$

satisfying the traveling-wave ordinary differential equation (ODE)

$$B(\bar{U})\bar{U}' = F(\bar{U}) - F(U_-). \tag{5.5}$$

Considering the block structure of B, this can be written as:

$$F^I(u^I, u^{II}) \equiv F^I(u^I_-, u^{II}_-) \tag{5.6}$$

and

$$b_1(u^I)' + b_2(u^{II})' = F^{II}(u^I, u^{II}) - F^{II}(u^I_-, u^{II}_-). \tag{5.7}$$

We assume that, by some invertible change of coordinates $U \to W(U)$, possibly but not necessarily connected with a global convex entropy, followed if necessary by multiplication on the left by a nonsingular matrix function $S(W)$, equations (5.1) may be written in the *quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde{A}^0 W_t + \tilde{A} W_x = (\tilde{B} W_x)_x + G, \quad W = \begin{pmatrix} w^I \\ w^{II} \end{pmatrix}, \tag{5.8}$$

$w^I \in \mathbb{R}^{n-r}$, $w^{II} \in \mathbb{R}^r$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, where, defining $W_\pm := W(U_\pm)$:
(A1) $\tilde{A}(W_\pm)$, $\tilde{A}_* := \tilde{A}_{11}$, $\tilde{A}^0$ are symmetric, $\tilde{A}^0 > 0$.

(A2) No eigenvector of $dF(U_\pm)$ lies in the kernel of $B(U_\pm)$. (Equivalently, no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(W_\pm)$ lies in the kernel of $\tilde{B}(W_\pm)$.)
(A3) $\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}$, $\tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$, with $Re\tilde{b}(W) \geq \theta$ for some $\theta > 0$, for all W,

and $\tilde{g}(W_x, W_x) = \mathcal{O}(|W_x|^2)$. Here, the coefficients of (5.8) may be expressed in terms of the original equation (5.1), the coordinate change $U \to W(U)$, and the approximate symmetrizer $S(W)$, as

$$
\begin{aligned}
\tilde{A}^0 &:= S(W)(\partial U/\partial W), \quad \tilde{A} := S(W)d(\partial U/\partial W), \\
\tilde{B} &:= S(W)B(\partial U/\partial W), \quad G = -(dSW_x)B(\partial U/\partial W)W_x.
\end{aligned}
\tag{5.9}
$$

For examples about Navier–Stokes and Magnetohydrodynamic equations, see [**Z.2**]. Along with the above structural assumptions, we make the technical hypotheses:
(H0) F, B, W, $S \in C^s$, with $s \geq 5$.

(H1) The eigenvalues of $\tilde{A}_*$ are (i) distinct from 0; (ii) of common sign; and (iii) of constant multiplicity with respect to U.
(H2) $\sigma(dF(U_\pm))$ real, distinct, and nonzero.
(H3) Local to $\bar{U}(\cdot)$, solutions of (5.4)–(5.5) form a smooth manifold $\{\bar{U}^\delta(\cdot)\}$, $\delta \in \mathcal{U} \subset \mathbb{R}^\ell$.

Analogous to lemma 4.1 and corollary 4.2 we have the following lemma proved in [**MaZ.3**].

Lemma 5.1. *Given (H1)–(H3), the endstates $U_\pm$ are hyperbolic rest points of the ODE determined by (5.7) on the r-dimensional manifold (5.6), i.e., the coefficients of the linearized equations about $U^\pm$, written in local coordinates, have no center subspace. In particular, under regularity (H0),*

$$
D_x^j D_\delta^i (\bar{U}^\delta(x) - U_\pm) = \mathbf{O}(e^{-\alpha|x|}), \quad \alpha > 0,\ 0 \leq j \leq 6,\ i = 0, 1,
\tag{5.10}
$$

as $x \to \pm\infty$.

We now recall some important ideas of Kawashima et al concerning the smoothing effects of hyperbolic–parabolic coupling. The following results assert that hyperbolic effects can compensate for degenerate viscosity B, as depicted by the existence of a *compensating matrix* K.

Lemma 5.2. *([**KSh**]) Assuming A^0, A, B symmetric, $A^0 > 0$, and $B \geq 0$, the genuine coupling condition (GC) No eigenvector of A lies in* $\ker B$ *is*

equivalent to either of:
(K1) There exists a smooth skew-symmetric matrix function $K(A^0, A, B)$ such that

$$\textit{Re}\ \big(K(A^0)^{-1}A + B\big)(U) > 0. \tag{5.11}$$

(K2) For some $\theta > 0$, there holds

$$\operatorname{Re} \sigma(-i\xi(A^0)^{-1}A - |\xi|^2(A^0)^{-1}B) \leq -\theta|\xi|^2/(1+|\xi|^2), \tag{5.12}$$

for all $\xi \in \mathbb{R}$.

Proof. These and other useful equivalent formulations are established in [**KSh**]; see also [**Z.3, MaZ.4, Z.2**]. □

Now returning to the original equation (5.1) with $\bar{U}$ a shock solution we linearize around $\bar{U}$ exactly as we did in section 4, and we define again A and B in the same manner:

$$B(x) := B(\bar{U}(x)), \quad A(x)V := dF(\bar{U}(x))V - dB(\bar{U}(x))V\bar{U}_x. \tag{5.13}$$

Assume for A and B the block structures:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

The characteristics speeds $a_i^\pm$, the left and right eigenvalues $l_i^\pm, r_i^\pm$ for $dF(u_\pm)$ and $\beta_i^\pm = l_i^\pm B^\pm r_i^\pm$ are all defined the same way as before and again we have $\beta_i^\pm > 0$.

Also, let $a_j^*(x)$, $j = 1, \ldots, (n-r)$ denote the eigenvalues of

$$A_* := A_{11} - A_{12}B_{22}^{-1}B_{21},$$

with $l_j^*(x)$, $r_j^*(x) \in \mathbb{R}^{n-r}$ associated left and right eigenvectors, normalized so that $l_j^{*t} r_j \equiv \delta_k^j$. More generally, for an m_j^*-fold eigenvalue, we choose $(n-r) \times m_j^*$ blocks L_j^* and R_j^* of eigenvectors satisfying the *dynamical normalization*

$$L_j^{*t}\partial_x R_j^* \equiv 0,$$

along with the usual static normalization $L_j^{*t}R_j \equiv \delta_k^j I_{m_j^*}$; as shown in Lemma 4.9, [**MaZ.1**], this may always be achieved with bounded L_j^*, R_j^*. Associated with L_j^*, R_j^*, define extended, $n \times m_j^*$ blocks

$$\mathcal{L}_j^* := \begin{pmatrix} L_j^* \\ 0 \end{pmatrix}, \quad \mathcal{R}_j^* := \begin{pmatrix} R_j^* \\ -B_{22}^{-1}B_{21}R_j^* \end{pmatrix}.$$

Eigenvalues a_j^* and eigenmodes $\mathcal{L}_j^*$, $\mathcal{R}_j^*$ correspond, respectively, to short-time hyperbolic characteristic speeds and modes of propagation for the reduced, hyperbolic part of degenerate system (5.1).

Define local, $m_j \times m_j$ *dissipation coefficients*

$$\eta_j^*(x) := -L_j^{*t} D_* R_j^*(x), \quad j = 1, \ldots, J \leq n - r,$$

where

$$D_*(x) :=$$
$$A_{12}B_{22}^{-1}\Big[A_{21} - A_{22}B_{22}^{-1}B_{21} + A_*B_{22}^{-1}B_{21} + B_{22}\partial_x(B_{22}^{-1}B_{21})\Big]$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman–Enskog expansion in the (dual) relaxation case.

At $x = \pm\infty$, these reduce to the corresponding quantities identified by Zeng [**Ze.1, LZe**] in her study by Fourier transform techniques of decay to *constant solutions* $(\bar{u}, \bar{v}) \equiv (u_\pm, v_\pm)$ of hyperbolic–parabolic systems, i.e., of limiting equations

$$U_t = L_\pm U := -A_\pm U_x + B_\pm U_{xx}.$$

As a consequence of dissipativity, (A2), we obtain (see, e.g., [**Kaw, LZe, MaZ.3**], or Lemma 5.11)

$$\beta_j^\pm > 0, \quad Re\sigma(\eta_j^{*\pm}) > 0 \quad \text{for all } j. \tag{5.14}$$

However, note that the dynamical dissipation coefficient $D_*(x)$ *does not* agree with its static counterpart, possessing an additional term $B_{22}\partial_x(B_{22}^{-1}B_{21})$, and so we cannot conclude that (5.14) holds everywhere along the profile, but only at the endpoints. This is an important difference in the variable-coefficient case; see Remarks 1.11-1.12 of [**MaZ.3**] for further discussion.

We also make the following assumptions, necessary for linear stability: Assumptions (D):

$(D1)$ L has no (L^2, without loss of generality) eigenvalues in $\{Re\lambda \geq 0\} \setminus \{0\}$.

$(D2)$ $\{r_j^\pm; a_j^\pm \gtrless 0\} \cup \{\int_{-\infty}^{+\infty} \frac{\partial \bar{U}^\delta}{\partial \delta_i} dx; i = 1, \cdots, \ell\}$ is a basis for $\mathbb{R}^n$, with $\int_{-\infty}^{+\infty} \frac{\partial \bar{U}^\delta}{\partial \delta_i} dx$ computed at $\delta = 0$.

Proposition 5.3. *[MaZ.3] Under assumptions (A1)–(A3), (H0)–(H3), and (D1)–(D2), the Green distribution $G(x,t;y)$ associated with the linearized evolution equations may be decomposed as*

$$G(x,t;y) = H + E + S + R,$$

where, for $y \le 0$:

$$\begin{aligned} H(x,t;y) &:= \sum_{j=1}^{J} a_j^{*-1}(x) a_j^*(y) \mathcal{R}_j^*(x) \zeta_j^*(y,t) \delta_{x-\bar{a}_j^* t}(-y) \mathcal{L}_j^{*t}(y) \\ &= \sum_{j=1}^{J} \mathcal{R}_j^*(x) \mathcal{O}(e^{-\eta_0 t}) \delta_{x-\bar{a}_j^* t}(-y) \mathcal{L}_j^{*t}(y), \end{aligned} \tag{5.15}$$

where the averaged convection rates $\bar{a}_j^* = \bar{a}_j^*(x,t)$ in (5.15) denote the time-averages over $[0,t]$ of $a_j^*(x)$ along backward characteristic paths $z_j^* = z_j^*(x,t)$ defined by

$$dz_j^*/dt = a_j^*(z_j^*), \quad z_j^*(t) = x,$$

and the dissipation matrix $\zeta_j^* = \zeta_j^*(x,t) \in \mathbb{R}^{m_j^* \times m_j^*}$ is defined by the dissipative flow

$$d\zeta_j^*/dt = -\eta_j^*(z_j^*)\zeta_j^*, \quad \zeta_j^*(0) = I_{m_j}.$$

E and S have exactly the same form as in proposition 4.4, and

(5.16)

$$\begin{aligned} R(x,t;y) &= \mathbf{O}(e^{-\eta(|x-y|+t)}) \\ &\quad + \sum_{k=1}^{n} \mathbf{O}\left((t+1)^{-1/2} e^{-\eta x^+} + e^{-\eta|x|} \right) t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} \\ &\quad + \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t| \ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}, \\ &\quad + \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t| \ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1/2}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}, \end{aligned}$$

(5.17)

$$\begin{aligned}
R_y(x,t;y) &= \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar{a}_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \\
&+ \sum_{k=1}^{n} \mathbf{O}\left((t+1)^{-1/2}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1} e^{-(x-y-a_k^- t)^2/Mt} \\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}((t+1)^{-1/2} t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-},
\end{aligned}$$

(5.18)

$$\begin{aligned}
R_x(x,t;y) &= \sum_{j=1}^{J} \mathbf{O}(e^{-\eta t})\delta_{x-\bar{a}_j^* t}(-y) + \mathbf{O}(e^{-\eta(|x-y|+t)}) \\
&+ \sum_{k=1}^{n} \mathbf{O}\left((t+1)^{-1}e^{-\eta x^+} + e^{-\eta|x|}\right) t^{-1}(t+1)^{1/2} e^{-(x-y-a_k^- t)^2/Mt} \\
&+ \sum_{a_k^->0,\, a_j^-<0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1}) e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt} e^{-\eta x^+} \\
&+ \sum_{a_k^->0,\, a_j^+>0} \chi_{\{|a_k^- t|\ge |y|\}} \mathbf{O}(t+1)^{-1/2} t^{-1}) e^{-(x-a_j^+(t-|y/a_k^-|))^2/Mt} e^{-\eta x^-}.
\end{aligned}$$

Moreover, for $|x-y|/t$ sufficiently large, $|G| \le Ce^{-\eta t}e^{-|x-y|^2/Mt)}$ as in the strictly parabolic case.

Once again let $\tilde{G} = S + R$ and define e_i's as before. Obviously the same bounds mentioned for S and e_i's in lemma 4.8 hold here also. Furthermore we have the following for H:

Lemma 5.4. *With the conditions in proposition 5.3, H satisfies:*

$$|\int_{-\infty}^{+\infty} H(\cdot,t;y)f(y)dy|_{L^p} \le Ce^{-\eta t}|f|_{L^p},$$

$$|\int_{-\infty}^{+\infty} H_x(\cdot,t;y)f(y)dy|_{L^p} \le Ce^{-\eta t}|f|_{W^{1,p}},$$

for some $C, \eta > 0$, for any $p \ge 1$ and $f \in W^{1,p}$.

Proof. See [**MaZ.3, MaZ.4**]. $\square$

From here on, almost everything would be very similar to the strictly parabolic case in section 4: by replacing $\bar{U}$ with $\bar{U}^{\delta_0}$, for a small δ_0, we may assume that the initial perturbation has no mass at the $\frac{\partial \bar{U}^\delta}{\partial \delta_i}$ directions, then we define diffusion waves, φ_i, exactly as in (4.31) and (4.32), then $\varphi = \sum_{a_i^- < 0} \varphi_i + \sum_{a_i^+ > 0} \varphi_i$, and once again $V = \tilde{U} - \bar{U}^{\delta_0} - \varphi - \frac{\partial \bar{U}^\delta}{\partial \delta}\delta(t)$ with δ to be found (from now on we once again assume, without loss of generality, $\delta_0 = 0$). The equalities (4.34) to (4.38) are reproduced exactly as before, but with u and v replaced by U and V, respectively. Furthermore, it is easy to see that

$$
\begin{aligned}
\mathcal{F}(V, \varphi, \frac{\partial \bar{U}^\delta}{\partial \delta}\delta(t))_x = \mathbf{O}\big(&\mathcal{F}(V, \varphi, \frac{\partial \bar{U}^\delta}{\partial \delta}\delta) \\
&+ |(V + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta)_x||(v^{II} + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta)_x| \\
&+ |V + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta||(v^{II} + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta)_{xx}|\big).
\end{aligned}
\tag{5.19}
$$

The function $\delta(t)$ is defined as in (4.49), and then, similar to (4.50), we have:

$$
\begin{aligned}
V(x,t) = &\int_{-\infty}^{+\infty} (H + \tilde{G})(x, t; y) V(y, 0) dy \\
&- \int_0^t \int_{-\infty}^{+\infty} (H + \tilde{G})(x, t-s; y) \mathcal{F}(\varphi, V, \frac{\partial \bar{u}^\delta}{\partial \delta}\delta)_y(y,s) dy\, ds, \\
&+ \int_0^t \int_{-\infty}^{+\infty} (H + \tilde{G})(x, t-s; y) \Psi(y,s) dy\, ds.
\end{aligned}
\tag{5.20}
$$

Theorem 5.5. *Let (A1)–(A3) and (H0)–(H3), (D1)–(D2) hold, and* $|U_0|_{L^1 \cap L^\infty \cap H^3}$, $|xU_0|_{L^1} \le E_0$, E_0 *sufficiently small. Assume the above setting and* $U = V + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta$*; then for any* ϵ, $0 < \epsilon < \frac{1}{8}$,

$$
|V(\cdot, t)|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-1/p) - \frac{1}{4}},
\tag{5.21}
$$

$$
|\delta(t)| \le CE_0(1+t)^{-\frac{1}{2}+\epsilon},
\tag{5.22}
$$

$$
|\dot{\delta}(t)| \le CE_0(1+t)^{-1+\epsilon},
\tag{5.23}
$$

for any p, $1 \le p \le \infty$, *and with* C *independent of* p *(but depending on* ϵ*).*

Remark 5.6. Corollaries similar to 4.15 and 4.16 are valid here also.

Proof. Fixing ϵ, define

(5.24)
$$\zeta(t) := \sup_{0\le s\le t, 1\le p\le\infty} |V(\cdot,s)|_{L^p}(1+s)^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{4}} + \sup_{0\le s\le t} |\delta(s)|(1+s)^{\frac{1}{2}-\epsilon} + \sup_{0\le s\le t} |\dot{\delta}(s)|(1+s)^{1-\epsilon}.$$

To show

$$\zeta(t) \le C(E_0 + \zeta^2(t)) \tag{5.25}$$

we need to show

$$|V(\cdot,s)|_{L^p} \le C(E_0+\zeta^2(t))(1+s)^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{4}} \tag{5.26}$$

and

$$|\delta(s)| \le C(E_0+\zeta^2(t))(1+s)^{-\frac{1}{2}+\epsilon} \tag{5.27}$$

and a similar statement for $\dot{\delta}$. From here on the proof goes very much similarly to the proof of theorem 4.14, except for two issues: first, here we do not have a lemma similar to lemma 4.13, as the short time estimates there need the strict parabolic hypothesis. Instead we have to use some energy estimates in order to control the derivatives of V. Using this method we will find out that, under the assumptions of the problem,

$$|V(\cdot,t)|_{H^3} \le C(E_0+\zeta(t))(1+s)^{-\frac{1}{2}}. \tag{5.28}$$

This in turn will implies (4.66), for $1 \le p \le p^*$, with p^* as before (see remark 4.18). The other difference is that, here we have the extra term H in the Green function decomposition, but we do not have any bounds for H_y. Hence we have to compute

$$\int_0^t \int_{-\infty}^{+\infty} H(x,t;y)\mathcal{F}_y(y,s)dyds.$$

By (5.19) and (5.28), we obtain

$$|\mathcal{F}_x(\cdot,s)|_{L^p} \le C(E_0+\zeta(t)^2)s^{-\frac{1}{2}(1-\frac{1}{p})-\frac{3}{4}},$$

$1 \le p \le p^*$. This with lemma 5.4 provides us with necessary bounds.

It remains to show that (5.28) holds. Let $\tilde{U} - \bar{U} = V + \varphi + \frac{\partial \bar{U}^\delta}{\partial \delta}\delta$, and $W := \tilde{W} - \bar{W} - \hat{\varphi} - \frac{\partial \bar{W}^\delta}{\partial \delta}\delta$, with $\hat{\varphi} = d\bar{W}\varphi$. Notice also that $d\bar{W}(\frac{\partial \bar{U}^\delta}{\partial \delta}) = \frac{\partial \bar{W}^\delta}{\partial \delta}$ (we use the notation $d\bar{W} = dW(\bar{U})$, $d\bar{U} = dU(\bar{W})$, etc).

Claim: $|V|_{H^r} \sim |W|_{H^r}$.
proof of the claim: Note that $\tilde{U} - \bar{U} = dU_{Ave}(\tilde{W} - \bar{W})$, where $dU_{Ave} = \int_0^1 dU(\bar{W} + \theta(\tilde{W} - \bar{W}))d\theta$. Now using the facts that $\frac{\partial \bar{U}^\delta}{\partial \delta} = d\bar{U}\frac{\partial \bar{W}^\delta}{\partial \delta}$ and $\varphi = d\bar{U}\hat{\varphi}$ we deduce:

$$V = dU_{Ave}(W) + (dU_{Ave} - d\bar{U})\hat{\varphi} + (dU_{Ave} - d\bar{U})\frac{\partial \bar{W}^\delta}{\partial \delta}\delta.$$

This, with a similar argument in the reverse direction proves our claim.

From here on we follow closely the argument presented by Zumbrun in **[Z.3]** (see also **[MaZ.4]**), with some necessary modification to handle our more complicated case, e.g., key cancelations in (5.38) and (5.39) and the term ξ in (5.48), which has no counterpart in **[Z.3]**. First, we introduce the weighted norms and inner product

$$|f|_\alpha := |\alpha^{1/2} f|_{L^2}, \quad |f|_{H^s_\alpha} := \sum_{r=0}^{s} |\partial_x^r f|_\alpha, \quad \langle f, g\rangle_\alpha := \langle \alpha f, g\rangle_{L^2}, \tag{5.29}$$

$\alpha(x)$ scalar, uniformly positive, and uniformly bounded. For the remainder of this section, we shall for notational convenience omit the subscript α, referring always to α-norms or -inner products unless otherwise specified. For later reference, we note the commutator relation

$$\langle f, g_x\rangle = -\langle f_x + (\alpha_x/\alpha) f, g\rangle, \tag{5.30}$$

and the related identities

$$\langle f, S f_x\rangle = -(1/2)\langle f, \big(S_x + (\alpha_x/\alpha)S\big) f\rangle, \tag{5.31}$$

$$\langle f, (Sf)_x\rangle = (1/2)\langle f, \big(S_x - (\alpha_x/\alpha)S\big) f\rangle, \tag{5.32}$$

valid for symmetric operators S.

By (H1)(ii), we have that $\bar{A}_{11}(\bar{A}^0_{11})^{-1}$ has real spectrum of uniform sign, without loss of generality negative, so that the similar matrix

$$(\bar{A}^0_{11})^{-1/2}\bar{A}_{11}(\bar{A}^0_{11})^{-1/2} = (\bar{A}^0_{11})^{-1/2}\bar{A}_{11}(\bar{A}^0_{11})^{-1}(\bar{A}^0_{11})^{1/2}$$

has real, negative spectrum as well. (Recall, $\bar{A}^0_{11}$ is symmetric negative definite as a principal minor of the symmetric negative definite matrix $\bar{A}^0$.) It follows that $\bar{A}_{11}$ itself is uniformly symmetric negative definite, i.e.,

$$\bar{A}_{11} \le -\theta < 0. \tag{5.33}$$

Defining α, following Goodman **[Go]**, by the ODE

$$\alpha_x = C_*|\bar{W}_x|\alpha, \quad \alpha(0) = 1, \tag{5.34}$$

where $C_* > 0$ is a large constant to be chosen later, we have by (5.33)

$$(\alpha_x/\alpha)\bar{A}_{11} \le -C_*\theta|\bar{W}_x|. \tag{5.35}$$

Note, because $|\bar{W}_x| \le Ce^{-\theta|x|}$, that α is indeed positive and bounded from both zero and infinity, as the solution of the simple scalar exponential growth equation (5.34).

Energy estimates for W:

$$\begin{aligned}
&\tilde{A}^0 W_t + \tilde{A}W_x - (\tilde{B}W_x)_x \\
&\qquad = -(\tilde{A} - \bar{A})\bar{W}_x + ((\tilde{B} - \bar{B})\bar{W}_x)_x \\
&\qquad\quad - \bar{A}\frac{\partial \bar{W}^\delta_x}{\partial \delta}\delta + \Big(\bar{B}\frac{\partial \bar{W}^\delta_x}{\partial \delta}\Big)_x \delta \\
&\qquad\quad - (\tilde{A} - \bar{A})\frac{\partial \bar{W}^\delta_x}{\partial \delta}\delta + \Big((\tilde{B} - \bar{B})\frac{\partial \bar{W}^\delta_x}{\partial \delta}\Big)_x \delta \\
&\qquad\quad - \tilde{A}^0\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta} - \tilde{A}^0\hat{\varphi}_t - \tilde{A}\hat{\varphi}_x + (\tilde{B}\hat{\varphi}_x)_x
\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}
&\tilde{A}^0 := A^0(\tilde{W}), \quad \tilde{A} := A(\tilde{W}), \quad \tilde{B} := B(\tilde{W}); \\
&\bar{A}^0 := A^0(\bar{W}), \quad \bar{A} := A(\bar{W}), \quad \bar{B} := B(\bar{W});
\end{aligned} \tag{5.37}$$

(notice that we dropped tilde signs from $\tilde{A}^0, \tilde{A}$ and $\tilde{B}$ in (5.9) and, with a slight abuse of notation, used them here differently). We want to write the right hand side of (5.36) in the form $\mathcal{M}_1 + (\mathcal{M}_2)_x + \xi$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ depending on W and "behaving well enough", and ξ is a remainder decaying fast enough.

Beginning from the last line in (5.36), we write

$$\tilde{A}^0\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta} = (\tilde{A}^0 - A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta} + A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta}$$

Now $A^0(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta})\frac{\partial\bar{W}^\delta}{\partial\delta}\dot{\delta}$ goes into ξ and $(\tilde{A}^0-A^0(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}))\frac{\partial\bar{W}^\delta}{\partial\delta}\dot{\delta}$ goes into $\mathcal{M}_1$. Notice that

$$\tilde{A}^0 - A^0(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta) = \int_0^1 dA^0(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta+\theta W)d\theta W$$

In a similar fashion $\tilde{A}^0\hat{\varphi}_t$ and $\tilde{A}\hat{\varphi}_x$ each gives rise to a term which goes into $\mathcal{M}_1$ and the other which goes into ξ. Similarly $(\tilde{B}\hat{\varphi}_x)_x$ comprises two terms, one of which is absorbed by $\mathcal{M}_2$ and the other by ξ.

Using

$$d\bar{A}\frac{\partial\bar{W}^\delta}{\partial\delta}\bar{W}_x + \bar{A}\frac{\partial\bar{W}^\delta_x}{\partial\delta} = (d\bar{B}\frac{\partial\bar{W}^\delta}{\partial\delta}\bar{W}_x)_x + (\bar{B}\frac{\partial\bar{W}^\delta_x}{\partial\delta})_x \tag{5.38}$$

we can write the second and third lines of (5.36) in the form:

$$\begin{aligned}
(\tilde{A}-\bar{A})\bar{W}_x &- ((\tilde{B}-\bar{B})\bar{W}_x)_x + \bar{A}\frac{\partial\bar{W}^\delta_x}{\partial\delta}\delta - (\bar{B}\frac{\partial\bar{W}^\delta_x}{\partial\delta})_x\delta \\
&= (\tilde{A}-A(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta))\bar{W}_x \\
&\quad + \left(A(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta) - \bar{A} - d\bar{A}(\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta)\right)\bar{W}_x + d\bar{A}\hat{\varphi}\bar{W}_x \\
&\quad - \left((\tilde{B}-B(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta))\bar{W}_x\right)_x \\
&\quad + \left((B(\bar{W}+\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta) - \bar{B} - d\bar{B}(\hat{\varphi}+\frac{\partial\bar{W}^\delta}{\partial\delta}\delta))\bar{W}_x\right)_x \\
&\quad - (d\bar{B}\hat{\varphi}\bar{W}_x)_x
\end{aligned} \tag{5.39}$$

the term in the second line of (5.39) goes into $\mathcal{M}_1$, the third line and the fifth lines go into ξ, and the fourth line goes into $\mathcal{M}_2$. The fourth line of (5.36) can be dealt with in a similar way.

To summarize, we were able to write equation (5.36) in the form:

$$\tilde{A}^0 W_t + \tilde{A}W_x - (\tilde{B}W_x)_x = \mathcal{M}_1 + (\mathcal{M}_2)_x + \xi(x,t) \tag{5.40}$$

where $\mathcal{M}_1, \mathcal{M}_2$ are dependent on W;

$$
\begin{aligned}
\mathcal{M}_1 =& -(\tilde{A}^0 - A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta} \\
& -(\tilde{A}^0 - A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\hat{\varphi}_t \\
& -(\tilde{A} - A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\hat{\varphi}_x \\
& -(\tilde{A} - A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\bar{W}_x \\
& -(\tilde{A} - A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\frac{\partial \bar{W}_x^\delta}{\partial \delta}\delta.
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
\mathcal{M}_2 =& (\tilde{B} - B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\hat{\varphi}_x \\
& + (\tilde{B} - B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\bar{W}_x \\
& + (\tilde{B} - B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\frac{\partial \bar{W}_x^\delta}{\partial \delta}\delta.
\end{aligned}
\tag{5.42}
$$

Using $\tilde{A} = \bar{A} + \mathcal{O}(\zeta)$, $\tilde{A}_x = \mathcal{O}(|\bar{W}_x| + \zeta)$ we can see:

$$
|\langle \partial_x^r W, \partial_x^r \mathcal{M}_1 \rangle| \leq C\langle \partial_x^r W, \bar{W}_x \partial_x^r W \rangle + C\zeta |W|_{H^r}^2 + C|W|_{H^{r-1}}^2
\tag{5.43}
$$

for $r = 0, \cdots, 3$. $\mathcal{M}_2$ has the block form: $\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M}_{22} \end{pmatrix}$, hence using Young's inequality and the block structure of $\mathcal{M}_2$,

$$
|\langle \partial_x^r W, \partial_x^r (\mathcal{M}_2)_x \rangle| \leq C\mu^{-1}|w^{II}|_{H^r}^2 + \mu |w^{II}|_{H^{r+1}}^2
\tag{5.44}
$$

for μ arbitrarily small. $\xi(x,t)$ is independent of W,

$$
\begin{aligned}
\xi = &- A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\frac{\partial \bar{W}^\delta}{\partial \delta}\dot{\delta} \\
&- A^0(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\hat{\varphi}_t \\
&- A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\hat{\varphi}_x \\
&+ (B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\hat{\varphi}_x)_x \\
&- \left(A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta) - \bar{A} - d\bar{A}(\hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta)\right)\bar{W}_x + d\bar{A}\hat{\varphi}\bar{W}_x \\
&- \left((B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta) - \bar{B} - d\bar{B}(\hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta))\bar{W}_x\right)_x \\
&+ (d\bar{B}\hat{\varphi}\bar{W}_x)_x \\
&- (A(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta) - \bar{A})\frac{\partial \bar{W}_x^\delta}{\partial \delta}\delta \\
&+ ((B(\bar{W} + \hat{\varphi} + \frac{\partial \bar{W}^\delta}{\partial \delta}\delta) - \bar{A})\frac{\partial \bar{W}_x^\delta}{\partial \delta})_x\delta.
\end{aligned} \tag{5.45}
$$

ξ has the good property that, with differentiation with respect to x, its rate of decay remains the same. Hence we transfer all the derivatives to ξ:

$$
|\langle D_x^r W, D_x^r \xi \rangle| \le C(|W|_{L^2}^2 + |\xi|_{H^{2r}}^2) \tag{5.46}
$$

Notice that under the assumptions and definitions of Theorem 5.5,

$$
|\xi(\cdot, s)|_{H^r} \le C(E_0 + \zeta(t))(1+s)^{-\frac{3}{4}} \tag{5.47}
$$

for any r and $0 \le s \le t$. Now the following lemma provides us with necessary bounds we need for $|W|_{H^3}$:

Lemma 5.7. *Under the hypotheses of Theorem 5.5 let $W_0 \in H^3$, and suppose that, for $0 \le t \le T$, both the supremum of $|\dot{\delta}|, |\delta|$ and the $W^{2,\infty}$ norm of the solution $W = (w^I, w^{II})^t$ remain bounded by a sufficiently small constant $\zeta > 0$. Then, for all $0 \le t \le T$,*

$$
|W(t)|_{H^3}^2 \le C|W(0)|_{H^3}^2 e^{-\theta t} + C\int_0^t e^{-\theta_2(t-\tau)}(|W|_{L^2}^2 + |\xi|_{H^6}^2)(\tau)\,d\tau. \tag{5.48}
$$

We first carry out a complete proof in the more straightforward case that the equations may be globally symmetrized , i.e., with conditions (A1)–(A3) replaced by the following global versions, indicating afterward by a few remarks the changes needed to carry out the proof in the general case.

(A1') $\tilde{A}^j$, $\tilde{A}^{jk}_* := \tilde{A}^{jk}_{11}$, $\tilde{A}^0$ are symmetric, $\tilde{A}^0 > 0$.

(A2') No eigenvector of $\sum \xi_j dF^j(U)$ lies in the kernel of $\sum \xi_j \xi_k B^{jk}(U)$, for all nonzero $\xi \in \mathbb{R}^d$. (Equivalently, no eigenvector of $\sum \xi_j \tilde{A}^j(\tilde{A}^0)^{-1}(W)$ lies in the kernel of $\sum \xi_j \xi_k \tilde{B}^{jk}(W)$.)

(A3')

$$\tilde{B}^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}, \tag{5.49}$$

with $Re \sum \xi_j \xi_k \tilde{b}^{jk}(W) \geq \theta |\xi|^2$ for some $\theta > 0$, for all W and all $\xi \in \mathbb{R}^d$, and $\tilde{G} \equiv 0$.

To prove (5.48), we carry out a series of successively higher order energy estimates of the type formalized by Kawashima [**Kaw**] and used extensively by K. Zumbrun et al (see [**Z.3**], also [**MaZ.3, MaZ.4**]) The origin of this approach goes back to [**Kan, MNi**] in the context of gas dynamics; see, e.g., [**HoZ.1**] for further discussion/references.

Let $\tilde{K}$ denote the skew-symmetric matrix described in Lemma 5.11 associated with $\tilde{A}^0$, $\tilde{A}$, $\tilde{B}$, satisfying

$$\tilde{K}(\tilde{A}^0)^{-1}\tilde{A} + \tilde{B} > 0.$$

Then, regarding $\tilde{A}^0$, $\tilde{K}$, we have

(5.50)

$$\begin{aligned} \tilde{A}^0_x &= dA^0(\tilde{W})\tilde{W}_x, \ \tilde{K}_x = dK(\tilde{W})\tilde{W}_x, \ \tilde{A}_x = dA(\tilde{W})\tilde{W}_x, \quad \tilde{B}_x = dB(\tilde{W})\tilde{W}_x, \\ \tilde{A}^0_t &= dA^0(\tilde{W})\tilde{W}_t, \ \tilde{K}_t = dK(\tilde{W})\tilde{W}_t, \quad \tilde{A}_t = dA(\tilde{W})\tilde{W}_t, \quad \tilde{B}_x = dB(\tilde{W})\tilde{W}_t. \end{aligned}$$

Now:

$$|\tilde{W}_x| = |W_x + \bar{W}_x + \hat{\varphi}_x + \frac{\partial \bar{W}^\delta_x}{\partial \delta}\delta| \leq |W_x| + |\bar{W}_x| + |\hat{\varphi}_x| + |\frac{\partial \bar{W}^\delta_x}{\partial \delta}\delta| \tag{5.51}$$

and, from the equation for $\tilde{A}$,

$$|\tilde{W}_t| \leq C(|\tilde{W}_x| + |\tilde{w}^{II}_{xx}|). \tag{5.52}$$

Thus, in particular it follows that

(5.53)
$$|\dot\delta|,\ |\tilde A^0_x|,\ |\tilde A^0_{xx}|,\ |\tilde K_x|,\ |\tilde K_{xx}|,\ |\tilde A_x|,\ |\tilde A_{xx}|,\ |\tilde B_x|,\ |\tilde B_{xx}|,\ |\tilde A^0_t|, |\tilde K_t|,\ |\tilde A_t|,\ |\tilde B_t| \le C(\zeta + |\bar U_x|).$$

In what follows, we shall need to keep careful track of the distinguished constant C_*.

Computing

$$-\langle W, \tilde A W_x\rangle = (1/2)\langle W, (\tilde A_x + (\alpha_x/\alpha)\tilde A)W\rangle \tag{5.54}$$

and expanding $\tilde A = \bar A + \mathcal{O}(\zeta)$, $\tilde A_x = \mathcal{O}(|\bar W_x| + \zeta)$, we obtain by (5.35) the key property

(5.55)
$$\begin{aligned}-\langle W, \tilde A W_x\rangle &= (1/2)\langle w^I, (\alpha_x/\alpha)\bar A_{11} w^I\rangle \\ &\quad + \mathcal{O}\Big(\langle |\bar W_x||W|, |W|\rangle + \langle (\alpha_x/\alpha)|W|, \zeta|W| + |w^{II}|\rangle\Big)\\ &\le -(C_*\theta/3)\langle |w^I|, |\bar W_x||w^I|\rangle + C\zeta|w^I|^2 + C(C_*)|w^{II}|^2,\end{aligned}$$

by which we shall control transverse modes, provided C_* is chosen sufficiently large, or, more generally,

(5.56)
$$-\langle \partial_x^k W, \tilde A\partial_x^k W_x\rangle \le -(C_*\theta/3)\langle |\partial_x^k w^I|, |\bar W_x||\partial_x^k w^I|\rangle + C\zeta|\partial_x^k w^I|^2 + C(C_*)|\partial_x^k w^{II}|^2.$$

Here and below, $C(C_*)$ denotes a suitably large constant depending on C_*, while C denotes a fixed constant independent of C_*: likewise, $\mathcal{O}(\cdot)$ indicates a bound independent of C_*.

Zeroth order "Friedrichs-type" estimate. We first perform a standard, zeroth- and first-order "Friedrichs-type" estimate for symmetrizable hyperbolic systems **[Fri]**. Taking the α-inner product of W against (5.40), we obtain after rearrangement, integration by parts using (5.30)–(5.31), and

several applications of Young's inequality, the energy estimate

$$
\begin{aligned}
\frac{1}{2}\langle W, \tilde A^0 W\rangle_t &= \langle W, \tilde A^0 W_t\rangle + \frac{1}{2}\langle W, \tilde A^0_t W\rangle \\
&= -\langle W, \tilde A W_x\rangle + \langle W, (\tilde B W_x)_x\rangle + \langle W, \mathcal{M}_1\rangle + \langle W, (\mathcal{M}_2)_x\rangle \\
&\quad + \langle W, \xi\rangle + \frac{1}{2}\langle W, \tilde A^0_t W\rangle \\
&= \frac{1}{2}\langle W, (\tilde A_x + (\alpha_x/\alpha)\tilde A) W\rangle - \langle W_x - (\alpha_x/\alpha)W, \tilde B W_x\rangle \\
&\quad + \langle W, \mathcal{M}_1\rangle - \langle W, \mathcal{M}_2\rangle + \langle W, \xi\rangle + \frac{1}{2}\langle W, \tilde A^0_t W\rangle \\
&\le -\langle W_x, \tilde B W_x\rangle + C(C_*)\int \alpha\Big((|W_x| + |\bar W_x|)|W|^2 \\
&\quad + |w^{II}_x||W|(|W_x| + |\bar W_x|) + C|W|^2_{L^2} + \mu|w^{II}|^2_{H^1} + |\langle W, \xi\rangle| \\
&\le -\theta|w^{II}|^2_{H^1} + C(C_*)\left(|W|^2_{L^2} + |\xi|^2_{L^2}\right).
\end{aligned}
\tag{5.57}
$$

Here, we used boundedness of $|\partial_x^r \bar W|$ and $|W_x|$ and also the inequalities (5.43), (5.44) and (5.46).

First order "Friedrichs-type" estimate. For first and higher derivative estimates, it is crucial to make use of the favorable terms (5.56) afforded by the introduction of α-weighted norms. Differentiating (5.40) with respect to x, taking the α-inner product of W_x against the resulting equation, and substituting the result into the first term on the righthand side of

$$
\frac{1}{2}\langle W_x, \tilde A^0 W_x\rangle_t = \langle W_x, (\tilde A^0 W_t)_x\rangle - \langle W_x, \tilde A^0_x W_t\rangle + \frac{1}{2}\langle W_x, \tilde A^0_t W_x\rangle,
\tag{5.58}
$$

we obtain after various simplifications and integrations by parts:

$$
\begin{aligned}
\frac{1}{2}\langle W_x, \tilde A^0 W_x\rangle_t &= -\langle W_x, (\tilde A W_x)_x\rangle + \langle W_x, (\tilde B W_x)_{xx}\rangle + \langle W_x, (\mathcal{M}_1)_x\rangle \\
&\quad + \langle W_x, (\mathcal{M}_2)_{xx}\rangle + \langle W_x, \xi_x\rangle \\
&\quad - \langle W_x, \tilde A^0_x W_t\rangle + \frac{1}{2}\langle W_x, \tilde A^0_t W_x\rangle \\
&= -\langle W_x, \tilde A W_{xx}\rangle - \langle W_x, \tilde A_x W_x\rangle \\
&\qquad - \langle W_{xx} + (\alpha_x/\alpha)W_x, \tilde B W_{xx} + \tilde B_x W_x\rangle \\
&\qquad + \langle W_x, (\mathcal{M}_1)_x\rangle - \langle W_x + (\mathcal{M}_2)_{xx}\rangle + \langle W_x, \xi_x\rangle \\
&\qquad - \langle W_x, \tilde A^0_x W_t\rangle + \frac{1}{2}\langle W_x, \tilde A^0_t W_x\rangle.
\end{aligned}
\tag{5.59}
$$

Estimating the first term on the righthand side of (5.59) using (5.56), $k = 1$, and substituting $(\tilde A^0)^{-1}$ times (5.40) into the second to last term on the

righthand side of (5.59), we obtain by (5.53) plus various applications of Young's inequality the next-order energy estimate:

(5.60)

$$
\begin{aligned}
\frac{1}{2}\langle W_x, \tilde A^0 W_x\rangle_t \le{}& -\langle W_x, \tilde A W_{xx}\rangle - \langle W_{xx}, \tilde B W_{xx}\rangle\\
&+ C(C_*)\langle |W_x^{II}| + \zeta|W_x|, (|W| + |W_x|)|\bar W_x| + |w_{xx}^{II}|\rangle\\
&+ C\langle (|W_x| + |w_{xx}^{II}|), |\bar W_x|(|W| + |W_x|)\rangle\\
&+ C\langle W_x, \bar W_x W_x\rangle + C\zeta |W_x|_{L^2}^2\\
&+ C\mu^{-1}|w_x^{II}|_{L^2}^2 + \mu|w_{xx}^{II}|_{L^2}^2 + C(C_*)(|W|_{L^2} + |\xi|_{H^2}^2)\\
\\
\le{}& -(\theta/2)|w_{xx}^{II}|_{L^2}^2 - (C_*\theta/4)\langle |\bar W_x||w_x^I|, |w_x^I|\rangle + C(C_*)\zeta|w_x^I|_{L^2}^2\\
&+ C(C_*)|w_x^{II}|_{L^2}^2 + C(C_*)(|W|_{L^2} + |\xi|_{H^2}^2),
\end{aligned}
$$

provided C_* is sufficiently large and ζ, μ sufficiently small.

First order "Kawashima-type" estimate. Next, we perform a "Kawashima-type" derivative estimate. Taking the α-inner product of W_x against $\tilde K(\tilde A^0)^{-1}$ times (5.40), and noting that (integrating by parts, and using skew-symmetry of $\tilde K$)

(5.61)

$$
\begin{aligned}
\frac{1}{2}\langle W_x, \tilde K W\rangle_t &= \frac{1}{2}\langle W_x, \tilde K W_t\rangle + \frac{1}{2}\langle W_{xt}, \tilde K W\rangle + \frac{1}{2}\langle W_x, \tilde K_t W\rangle\\
&= \frac{1}{2}\langle W_x, \tilde K W_t\rangle - \frac{1}{2}\langle W_t, \tilde K W_x\rangle\\
&\quad - \frac{1}{2}\langle W_t, \big(\tilde K_x + (\alpha_x/\alpha)\big)W\rangle + \frac{1}{2}\langle W_x, \tilde K_t W\rangle\\
&= \langle W_x, \tilde K W_t\rangle + \frac{1}{2}\langle W, \big(\tilde K_x + (\alpha_x/\alpha)\big)W_t\rangle + \frac{1}{2}\langle W_x, \tilde K_t W\rangle,
\end{aligned}
$$

we obtain by calculations similar to the above the auxiliary energy estimate:

$$
\begin{aligned}
\frac{1}{2}\langle W_x, \tilde K W\rangle_t \le{}& -\langle W_x, \tilde K(\tilde A^0)^{-1}\tilde A W_x\rangle\\
&+ C(C_*)|w_x^{II}|^2 + C\langle (|\bar W_x| + \bar\zeta + \zeta)|w_x^I|, |w_x^I|\rangle\\
&+ C\bar\zeta^{-1}|w_{xx}^{II}|^2 + C(C_*)(|W|_{L^2} + |\xi|_{H^1}^2),
\end{aligned}
\tag{5.62}
$$

where $\bar\zeta > 0$ is an arbitrary constant arising through Young's inequality. (Here, we have estimated term $\langle \tilde A U_x, (\alpha_x/\alpha)U\rangle$ arising in the middle term of the righthand side of (5.61) using (5.31) by $C(C_*)\int |\bar W_x||U|^2 \le C(C_*)|U|_{L^\infty}^2$.)

Combined, weighted H^1 estimate. Choosing $\zeta << \bar\zeta << 1$, adding (5.62) to the sum of (5.60) times a suitably large positive constant $M(C_*,\bar\zeta) >> \bar\zeta^{-1}$, and (5.57) times $M(C_*,\bar\zeta)^2$ and recalling 5.11, we obtain, finally, the combined first-order estimate

$$\begin{aligned}&\frac{1}{2}\Big(M(C_*,\bar\zeta)^2\langle W,\tilde A^0W\rangle+\langle W_x,\tilde KW\rangle+M(C_*,\bar\zeta)\langle W_x,\tilde A^0W_x\rangle\Big)_t\\ &\qquad\le -\theta(|W_x|_{L^2}^2+|w_{xx}^{II}|_{L^2}^2)+C(C_*)\left(|W|_{L^2}^2+|\xi|_{H^1}^2\right),\end{aligned}\tag{5.63}$$

$\theta>0$, for any $\bar\zeta$, $\zeta(\bar\zeta,C_*)$ sufficiently small, and C_*, $C(C_*)$ sufficiently large.

Higher order estimates. Performing the same procedure on the twice- and thrice-differentiated versions of equation (5.40), we obtain, likewise, Friedrichs estimates

(5.64)
$$\begin{aligned}\frac{1}{2}\langle\partial_x^qW,\tilde A^0\partial_x^qW\rangle_t&\le-(\theta/2)|\partial_x^{q+1}w^{II}|_{L^2}^2-(C_*\theta/4)\langle|\bar W_x||\partial_x^qw^I|,|\partial_x^qw^I|\rangle\\ &\quad+C(C_*)\big(\zeta|\partial_x^qw^I|_{L^2}^2+|\partial_x^qw^{II}|_{L^2}^2+|W_x|_{H_\alpha^{q-2}}+|W|_{L^2}^2+|\xi|_{H^{2q}}^2\big),\end{aligned}$$

and Kawashima estimates

(5.65)
$$\begin{aligned}\frac{1}{2}\langle\partial_x^qW,\tilde K\partial_x^{q-1}W\rangle_t&\le-\langle\partial_x^qW,\tilde K(\tilde A^0)^{-1}\tilde A\partial_x^qW\rangle\\ &\quad+C(C_*)|\partial_x^qw^{II}|_{L^2}^2+C\langle(|\bar W_x|+\bar\zeta+\zeta)|\partial_x^qw^I|,|\partial_x^qw^I|\rangle\\ &\quad+C\bar\zeta^{-1}|\partial_x^{q+1}w^{II}|_{L^2}^2+C(C_*)(|W_x|_{H_\alpha^{q-2}}+|W|_{L^2}^2+|\xi|_{H^{2q}}^2),\end{aligned}$$

for $q=2$, 3, provided $\bar\zeta$, $\zeta(\bar\zeta,C_*)$ are sufficiently small, and C_*, $C(C_*)$ are sufficiently large. The calculations are similar to those carried out already; see also the closely related calculations of Appendix A, **[MaZ.2].**

Final estimate. Adding $M(C_*,\bar\zeta)^2$ times (5.63), $M(C_*,\bar\zeta)$ times (5.64), and (5.65), with $q=2$, where M is chosen still larger if necessary, we obtain

(5.66)
$$\begin{aligned}&\frac{1}{2}\Big(M(C_*,\bar\zeta)^4\langle W,\tilde A^0W\rangle+M(C_*,\bar\zeta)^2\langle W_x,\tilde KW\rangle+M(C_*,\bar\zeta)^3\langle W_x,\tilde A^0W_x\rangle\\ &\qquad+\langle\partial_x^2W,\tilde K\partial_xW\rangle+M(C_*,\bar\zeta)\langle\partial_x^2W,\tilde A^0\partial_x^2W\rangle\Big)_t\\ &\quad\le-\theta(|W_x|_{H_\alpha^1}^2+|w_x^{II}|_{H_\alpha^2}^2)+C(C_*)\left(|W|_{L^2}^2)+|\xi|_{H^4}^2\right).\end{aligned}$$

Adding now $M(C_*,\bar\zeta)^2$ times (5.67), $M(C_*,\bar\zeta)$ times (5.64), and (5.65),

with $q=3$, we obtain the final higher-order estimate
(5.67)

$$\begin{aligned}
&\frac{1}{2}\Big(M(C_*,\bar\zeta)^6\langle W,\tilde A^0 W\rangle + M(C_*,\bar\zeta)^4\langle W_x,\tilde K W\rangle + M(C_*,\bar\zeta)^5\langle W_x,\tilde A^0 W_x\rangle \\
&\qquad + M(C_*,\bar\zeta)^2\langle \partial_x^2 W,\tilde K\partial_x W\rangle + M(C_*,\bar\zeta)^3\langle \partial_x^2 W,\tilde A^0\partial_x^2 W\rangle \\
&\qquad + \langle \partial_x^3 W,\tilde K\partial_x^2 W\rangle + M(C_*,\bar\zeta)\langle \partial_x^3 W,\tilde A^0\partial_x^3 W\rangle\Big)_t \\
&\quad \le -\theta(|W_x|^2_{H^2_\alpha} + |w_x^{II}|^2_{H^2_\alpha}) + C(C_*)\left(|W|^2_{L^2} + |\xi|^2_{H^6}\right). \\
&\quad \le -\theta|W|^2_{H^3_\alpha} + C(C_*)\left(|W|^2_{L^2}) + |\xi|^2_{H^6}\right).
\end{aligned}$$

Denoting

(5.68)

$$\begin{aligned}
\mathcal{E}(W) := \frac{1}{2}\Big(&M(C_*,\bar\zeta)^6\langle W,\tilde A^0 W\rangle + M(C_*,\bar\zeta)^4\langle W_x,\tilde K W\rangle \\
&+M(C_*,\bar\zeta)^5\langle W_x,\tilde A^0 W_x\rangle + M(C_*,\bar\zeta)^2\langle \partial_x^2 W,\tilde K\partial_x W\rangle + \\
M(C_*,\bar\zeta)^3\langle \partial_x^2 W,\tilde A^0\partial_x^2 W\rangle &+ \langle \partial_x^3 W,\tilde K\partial_x^2 W\rangle + M(C_*,\bar\zeta)\langle \partial_x^3 W,\tilde A^0\partial_x^3 W\rangle\Big),
\end{aligned}$$

we have by Young's inequality that $\mathcal{E}^{1/2}$ is equivalent to norms H^3 and H^3_α, hence (5.63) yields

$$\mathcal{E}_t \le -\theta_2\mathcal{E} + C(C_*)\left(|W|^2_{L^2}) + |\xi|^2_{H^6}\right),$$

from which we conclude,

$$\mathcal{E}(t) \le e^{-\theta_2 t}\mathcal{E}(0) + C(C_*)\int_0^t e^{-\theta_2(t-s)}\left(|W|^2_{L^2}) + |\xi|^2_{H^6}\right)(s)\,ds.$$

This is equivalent to (5.48).

The general case. It remains only to discuss the general case that hypotheses (A1)–(A3) hold as stated and not everywhere along the profile, with $\tilde G$ possibly nonzero. These generalizations requires only a few simple observations. The first is that we may express matrix $\tilde A$ in (5.40) as

$$\tilde A = \hat A + (|\bar W_x| + \zeta)\begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix},$$

where $\hat A$ is a symmetric matrix obeying the same derivative bounds as described for $\tilde A$, identical to $\tilde A$ in the 11 block and obtained in other blocks jk by smoothly interpolating over a bounded interval $[-R,+R]$ between

$\bar{A}(\tilde{W}_{-\infty})_{jk}$ and $\bar{A}(\tilde{W}_{+\infty})_{jk}$. Replacing $\tilde{A}$ by $\hat{A}$ in the qth order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as (integrating by parts if necessary)

$$\langle \partial_x^q \mathcal{O}(|\bar{W}_x| + \zeta)|W|, |\partial_x^{q+1} w^{II}|\rangle$$

plus lower-order terms, hence absorbed using Young's inequality to recover the same Friedrichs-type estimates obtained in the previous case. Thus, we may relax (A1') to (A1).

The second observation is that, because of the favorable terms

$$-(C_*\theta/4)\langle |\bar{W}_x||\partial_x^q w^I|, |\partial_x^q w^I|\rangle$$

occurring in the righthand sides of the Friedrichs-type estimates, we need the Kawashima-type bound only to control the contribution to $|\partial_x^q w^I|^2$ coming from x near $\pm\infty$; more precisely, we require from this estimate only a favorable term

$$-\theta\langle \big(1 - \mathcal{O}(|\bar{W}_x| + \zeta + \bar{\zeta})\big)|\partial_x^q w^I|, |\partial_x^q w^I|\rangle$$

rather than $-\theta|\partial_x^q w^I|^2$ as in (5.62) and (5.65). But, this may easily be obtained by substituting for $\tilde{K}$ a skew-symmetric matrix-valued function $\hat{K}$ defined to be identically equal to $\bar{K}(+\infty)$ and $\bar{K}(-\infty)$ for $|x| > R$, and smoothly interpolating between $\bar{K}(\pm\infty)$ on $[-R, +R]$, and using the fact that

$$\big(\bar{K}(\bar{A}^0)^{-1}\bar{A} + \bar{B}\big)_{\pm} \geq \theta > 0,$$

hence

$$\big(\hat{K}(\tilde{A}^0)^{-1}\tilde{A} + \tilde{B}\big) \geq \theta(1 - \mathcal{O}(|\bar{W}_x| + \zeta)).$$

Thus, we may relax (A2') to (A2).

Finally, notice that the term $\tilde{G} - \bar{G}$ in the perturbation equation may be Taylor expanded as

$$\begin{pmatrix} 0 \\ \tilde{g}(\tilde{W}_x, \bar{U}_x) + g(\bar{W}_x, \tilde{W}_x) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix}$$

The first, linear term on the righthand side may be grouped with term $\tilde{A}^0 W_x$ and treated in the same way, since it decays at plus and minus spatial infinity and vanishes in the 1-1 block. The $(0, \mathcal{O}(|W_x|^2)$ nonlinear term may be treated as other source terms in the energy estimates Specifically, the worst-case terms $\langle \partial_x^3 W, K\partial_x^2 \mathcal{O}(|W_x|^2)\rangle$ and $\langle \partial_x^3 W, \partial_x^3(0, \mathcal{O}(|W_x|^2))\rangle = \langle \partial_x^4 w^{II}, \partial_x^2 \mathcal{O}(|W_x|^2)\rangle$ may be bounded, respectively, by $|W|_{W^{2,\infty}}|W|_{H^3}^2$ and $|W|_{W^{2,\infty}}|w^{II}|_{H^4}|W|_{H^3}$. Thus, we may relax (A3') to (A3), completing the proof of the general case (A1)–(A3) and the theorem. $\square$

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